

Folding defect affine Toda field theories

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Abstract

A folding process is applied to fused $a_r^{(1)}$ defects to construct defects for the non-simply laced affine Toda field theories of $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ at the classical level. Support for the hypothesis that these defects are integrable in the folded theories is provided by the observation that transmitted solitons retain their form. Further support is given by the demonstration that energy and momentum are conserved.

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1 Introduction

The affine Toda field theories (ATFTs) have seen an upsurge of interest due to the discovery of integrable defects. Thus far only for the $a_r^{(1)}$ [1, 2] and $a_2^{(2)}$ [3] models do integrable defects exist in the literature- even at the classical level- despite the search for defects in ATFT having been initiated a decade ago [1]. Compare this to the discovery of solitons in ATFT, where the construction of $a_r^{(1)}$ and $d_4^{(1)}$ solitons [4] was rapidly followed by solitons in the other models [5]. The different ATFTs have similar properties (e.g., they all possess soliton solutions [5]) so it is expected that defects should exist for all of the ATFTs- as such, an overarching goal in this field is to find and investigate the properties of all of the possible defects.

Folding, or reduction, is a powerful tool which allows properties of non-simply laced theories to be found using properties of simply laced theories¹ which are often easier to work with. Indeed, folding has previously been put to use to find, in the non-simply laced theories, solitons [5, 6] and also scattering matrices at the classical [7] and quantum [8] levels. Note that folding has not previously been applied to the study of defect ATFTs.

In this paper, a folding process, originally described in [9], is used to obtain candidate integrable defects for the $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ series of non-simply laced ATFTs by making use of $a_r^{(1)}$ solitons and defects. Two methods, which turn out to be linked, are used to suggest that these defects should be classically integrable. Firstly, what happens when a soliton is sent through the defect; secondly, in what circumstances does the defect conserve energy and momentum. An interesting by-product of the folding process is that it allows for the construction of multisolitons, breathers and fusing rules for these folded theories ($c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$)- something not explicitly considered previously in the literature.

What this paper does not do is provide defects for any of the ATFTs which do not fall under the umbrella of $a_r^{(1)}$, though it does suggest that if the appropriate defects can be found for the other simply laced theories ($d_s^{(1)}$, $e_6^{(1)}$, $e_7^{(1)}$ and $e_8^{(1)}$) then folding may be applied to give the defects of the other non-simply laced theories. Note that only one type of defect is given for each of $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ - there is no claim that other defects should not exist in these theories.

In order to set notation, and inform the reader unfamiliar with ATFT, a short summary of ATFT with defects is given below.

1.1 Affine Toda field theory and classical defects

To each affine Dynkin diagram there is a corresponding affine Toda model [10]. In 1+1 dimensions a working definition of ATFT is given by the Lagrangian

$$\mathcal{L}(u) = \frac{1}{2} \dot{u} \cdot \dot{u} - \frac{1}{2} u' \cdot u' - U(u) \quad (1.1)$$

¹The simply laced ATFTs are the $a_r^{(1)}$, $d_s^{(1)}$, $e_6^{(1)}$, $e_7^{(1)}$ and $e_8^{(1)}$ theories. They are distinguished in having all roots of the same length, conventionally $\sqrt{2}$; although in the a_1 case the choice of $\alpha = 1$ is more conventional. The non-simply laced theories all have more than one length of root present.

where u is an r component vector living in the root space described by the affine Dynkin diagram and is a Lorentz scalar. The potential is given by [4, 5]

$$U(u) = \frac{m^2}{\beta^2} \sum_{j=0}^r n_j \left(e^{\beta \alpha_j \cdot u} - 1 \right). \quad (1.2)$$

In (1.2), $\{\alpha_i\}$ ($i = 1, \dots, r$), are the positive simple roots of the root space while $\alpha_0 = -\sum_{j=1}^r n_j \alpha_j$ is the lowest root of the root space, corresponding to the extra node of the affine Dynkin diagram. It is the case that $\sum_{j=0}^r n_j \alpha_j = 0$, so conventionally $n_0 = 1$, while the other marks $\{n_j\}$ are characteristic of the underlying Lie algebra. The constant m sets a mass scale, which has no importance for the classical discussions herein so will be set to unity, $m = 1$. β is the coupling constant. The potential here is chosen such that the zero solution has zero energy.

The Euler-Lagrange equations applied to (1.1) give

$$\ddot{u} - u'' = -U_u \quad (1.3)$$

where U_u denotes the gradient of the potential with respect to the vector u .

Guided by the Lagrangian description of a sine-Gordon defect [1]; Bowcock, Corrigan and Zambon suggested an Ansatz for defects in the other ATFTs in [2] (subsequently referred to as a type I defect [3])

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x) \left(\frac{1}{2}uA\dot{u} + uB\dot{v} + \frac{1}{2}vC\dot{v} - D(u, v) \right). \quad (1.4)$$

Equation (1.4) describes a defect situated at $x = 0$ with A, B and C constant matrices. \mathcal{L}_u and \mathcal{L}_v are ‘bulk’ Lagrangians of the form (1.1) for the fields u and v respectively. In practice u and v always correspond to the same root data.

The Euler-Lagrange equations applied to (1.4) give at $x = 0$ the conditions

$$\begin{aligned} u' &= A\dot{u} + B\dot{v} - D_u \\ v' &= -C\dot{v} + B^T\dot{u} + D_v \end{aligned} \quad (1.5)$$

as well as the bulk equations of motion.

In [2] a defect Lax pair, taking into account (1.5), was constructed and it was shown that only the $a_r^{(1)}$ models may incorporate such a defect and remain integrable. It was subsequently shown in [11] that (1.4) conserves modified energy and momentum only for $a_r^{(1)}$ and that the conditions imposed are the same as for the Lax pair- i.e., in this case energy-momentum conservation implies integrability. From either approach it is found that

$$\begin{aligned} A &= C = 1 - B \\ D(u, v) &= d \sum_{j=0}^r e^{\frac{1}{2}\alpha_j \cdot (B^T u + Bv)} + d^{-1} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j \cdot B(u-v)} \end{aligned} \quad (1.6)$$

and so the defect can be classified by B along with the ‘rapidity’ parameter d . There are two possibilities for B , which are

$$B = \sum_{j=1}^r (\lambda_j - \lambda_{j+1}) \lambda_j^T \quad (1.7)$$

and its transpose

$$B = \sum_{j=1}^r (\lambda_j - \lambda_{j-1}) \lambda_j^T \quad (1.8)$$

where $\{\lambda_i\}$ are the fundamental weights of $a_r^{(1)}$, defined by $\lambda_i \cdot \alpha_j = \delta_{ij}$ for $i, j = 1, \dots, r$ and $\lambda_0 = 0$. In this paper B will be taken to mean (1.7), with B^T used if the defect is classified by (1.8).

In [2] the effect of the defect on a one soliton solution was examined. Using a one soliton Ansatz (of species p , say) for u and for v revealed that a B defect gives v a ‘time delay’ of

$$\Lambda_p = \frac{ie^\theta + d\omega^{\frac{p}{2}}}{ie^\theta + d\omega^{-\frac{p}{2}}} \quad (1.9)$$

while a B^T defect gives a time delay of

$$\tilde{\Lambda}_p = \frac{ie^\theta - \tilde{d}\omega^{-\frac{p}{2}}}{ie^\theta - \tilde{d}\omega^{\frac{p}{2}}} \quad (1.10)$$

where θ is the rapidity of the soliton and $\omega = e^{\frac{2\pi i}{r+1}}$. It should be noted that the time delay is dependent on the species of soliton. In particular, the species p soliton and the species $h - p$ soliton receive different delays, meaning that neither B nor B^T type I defects are compatible with the folding considered in this paper.

The framework for defects was extended in [3] to include integrable defects for $a_2^{(2)}$ by means of what is referred to as a type II defect with an Ansatz of the form

$$\mathcal{L} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)(uv - 2(u-v)\dot{\chi} - D(u, v, \chi)) \quad (1.11)$$

where u and v are now scalar fields in either a_1 or $a_2^{(2)}$ (the form of D depends upon which theory is being looked at) while χ is a scalar field which only exists at the defect, known as an auxiliary field.

It was also noticed that a system containing a type I B defect and a type I B^T defect with the same parameter d in $a_2^{(1)}$ would give the same overall time delay to the species 1 and species 2 solitons². As such, solitons possessing the $a_2^{(2)}$ symmetry in the field u would also possess it in the field v . This opens up the possibility that (1.11) for $a_2^{(2)}$ somehow arises from fusing two type I $a_2^{(1)}$ defects. This possibility is formulated and generalised to all ATFTs obtainable by folding $a_r^{(1)}$ in this paper.

1.2 Layout of this paper

Section 2 describes the foldings of $a_r^{(1)}$ which will be used in this paper, which may be referred to as *bivalent* foldings since the roots of the folded theories are obtained by identifying roots of $a_r^{(1)}$ pairwise (contrast with e.g., $d_4^{(1)} \rightarrow a_2^{(2)}$ by which one of the folded roots is obtained

²P. Bowcock, private communication. Also mentioned in [3]

by identifying four of the roots of $d_4^{(1)}$). Such foldings were also considered in [9]. Section 3 describes how solitons of the folded theories may be obtained from $a_r^{(1)}$ solitons via these folding processes. The relevant Hirota tau functions [12] are obtained. Where the folding being done is non-canonical (in the sense of [9], i.e., foldings not found in [13]) the tau functions are compared to those found in [6], which were obtained by canonical folding of $d_s^{(1)}$ models. Section 4 describes a fused $a_r^{(1)}$ defect and its effects on solitons and energy and momentum conservation. Section 5 considers folded defects and how they affect the folded solitons. Conservation of modified energy and momentum is then proven for the folded defects. This section is perhaps the most important, and opens up a number of possible future research directions. Section 6 gives the conclusions and outlook for this discourse. The appendices contain some additional calculations, including some explicit calculations in $a_2^{(2)}$.

2 Bivalent foldings of $a_r^{(1)}$

In this section the folding process used in later sections is formulated. It was noted in [9] that numerous foldings of this type are possible for a given $a_r^{(1)}$ and whilst other ways to fold $a_r^{(1)}$ are considered in [9] and [14], no extra folded theories arise in this way. Here the bivalent folding is done explicitly by means of a parameter $k \in \mathbb{Z}$.

This paper places its foundations in the firm ground of $a_r^{(1)}$, so it is useful to first state some of the properties of $a_r^{(1)}$. The roots of $a_r^{(1)}$ obey

$$\alpha_i \cdot \alpha_j = 2\delta_{ij} - \delta_{i(j+1)} - \delta_{i(j-1)} \quad (2.1)$$

for $i, j = 0, \dots, r$. It is useful to extend this range for i and j by identifying the indices modulo the Coxeter number h . For $a_r^{(1)}$, $h = r + 1$, so e.g., $\alpha_{-2} \equiv \alpha_{r-1}$. For $a_r^{(1)}$ the roots conventionally all have length $\sqrt{2}$ so the Cartan matrix is given by

$$C_{ij} = \frac{2}{\alpha_j \cdot \alpha_j} \alpha_i \cdot \alpha_j = \alpha_i \cdot \alpha_j$$

i.e., (2.1). Note that there is no sum over j . The eigenvalues of the Cartan matrix play an important rôle in the construction of solitons.

Out of the $r + 1$ roots $\{\alpha_0, \dots, \alpha_r\}$ any choice of r of them form a basis for the root space. Conventionally, the fundamental weights $\{\lambda_i\}$ are given by

$$\lambda_i \cdot \alpha_j = \delta_{ij}, \quad i, j = 1, \dots, r$$

with $\lambda_0 = 0$ so they form a dual basis to $\{\alpha_1, \dots, \alpha_r\}$. However, it is equally valid to define a set of fundamental weights by e.g.,

$$\lambda_i \cdot \alpha_j = \delta_{ij}, \quad i, j = 0, 1, \dots, q-1, q+1, \dots, r$$

with $\lambda_q = 0$. Note then that in (1.7) and its transpose the sum may be extended to include $j = 0$ and then whichever set of fundamental weights is used B and B^T remain the same as before. Note also that the relation

$$\alpha_i = 2\lambda_i - \lambda_{i-1} - \lambda_{i+1}$$

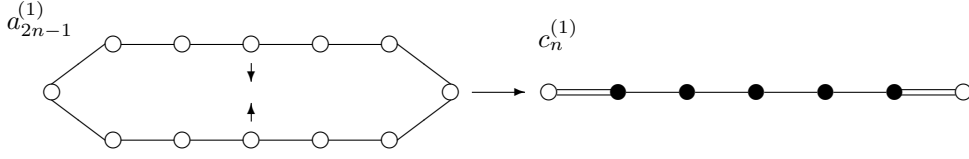


Figure 1: $a_{2n-1}^{(1)} \rightarrow c_n^{(1)}$.

is also valid for any choice of $\{\lambda_i\}$.

The affine Toda field, u , lives on the $a_r^{(1)}$ root space so a natural basis is given by the simple roots $\{\alpha_1, \dots, \alpha_r\}$: $u = u_1\alpha_1 + \dots + u_r\alpha_r$. However, one can consider an extended representation

$$u = \alpha_0 u_0 + \alpha_1 u_1 + \dots + \alpha_r u_r$$

where only r of the $r + 1$ components are independent. Noting then that $u_i = \lambda_i \cdot u$, it is reasonable to consider $u_q = 0$ for some $q \neq 0$ whilst retaining u_0 . This innocuous observation plays a rôle when considering the folding here in its full generality.

2.1 The bivalent foldings

The foldings considered here are bivalent in the sense that the roots of the folded theory will be obtained by identifying all roots of $a_r^{(1)}$ pairwise. Specifically, the roots α_i and α_{h+k-i} in $a_r^{(1)}$ will be paired up for all i . Which folded theory is obtained depends upon whether h is even (r odd) or odd (r even) and also on whether k is even or odd. That there are a number of distinct ways to identify the roots whilst obtaining a folded theory was already noted in [9]. Roots in the folded theories will be labelled by α' .

2.1.1 Case 1: h even, k even

This case includes the only canonical folding in this paper (i.e., the only one also found in [13]) when $k = 0$, that is $a_{2n-1}^{(1)} \rightarrow c_n^{(1)}$ ($h = 2n$). This folding is illustrated by figure 1. White nodes correspond to roots of length $\sqrt{2}$ and black nodes length 1. The nodes in the figure 1 have purposely not been labelled. For the $a_{2n-1}^{(1)}$ picture α_0 could really be placed anywhere, though conventionally it would be placed on the left. To go from a to c diagrammatically the nodes that are directly above/below each other are identified.

Here two nodes are self-identified, so there exist two solutions to $\alpha_i = \alpha_{h+k-i}$; the self-identified nodes are $\alpha_{\frac{k}{2}}$ and $\alpha_{\frac{h+k}{2}} = \alpha_{\frac{k}{2}+n}$. Upon folding the labelling of nodes is no longer democratic and to get the right Cartan matrix for the folded theory, the extra root α'_0 must be identified with one of these self-identified nodes. The choice made here is

$$\alpha'_0 = \alpha_{\frac{k}{2}}$$

implying that

$$\alpha'_i = \frac{\alpha_{\frac{k}{2}+i} + \alpha_{\frac{k}{2}+2n-i}}{2}.$$

(i, j)	$\alpha'_i \cdot \alpha'_j$
$(0, 0)$	2
$(s, s) \ s \neq 0, n$	1
(n, n)	2
$(s, s+1) \ s \neq 0, n-1$	$-\frac{1}{2}$
$(0, 1)$	-1
$(n-1, n)$	-1

Table 1: Inner products of $c_n^{(1)}$ root space.

For the $a_{2n-1}^{(1)}$ field u fold by identifying $u_{\frac{k}{2}} = 0$; $u_{\frac{k}{2}+i} = u_{\frac{k}{2}+2n-i} = \frac{\phi_i}{2}$ for $i = 1, \dots, n-1$ and $u_{\frac{k}{2}+n} = \phi_n$. So

$$u = \sum_{j=0}^{2n-1} u_j \alpha_j \longrightarrow \phi = \sum_{j=0}^n \phi_j \frac{\alpha_{\frac{k}{2}+j} + \alpha_{\frac{k}{2}+2n-j}}{2}.$$

In ϕ above, note that when $j = \frac{n}{2} = n$, both a_{2n-1} roots there are the same, so that term is merely $\phi_n \alpha_{n+\frac{k}{2}}$. The notation is that u is an unfolded field, and ϕ is a folded $c_n^{(1)}$ field, so

$$\phi = \sum_{j=0}^n \phi_j \alpha'_j.$$

It is readily seen in table 1 that this folding gives the correct $c_n^{(1)}$ inner products and so the kinetic terms of the Lagrangian (1.1) as required when folding $u \rightarrow \phi$. The $a_{2n-1}^{(1)}$ potential may be written as

$$U = e^{u \cdot \alpha_{\frac{k}{2}}} + \sum_{j=1}^{n-1} \left(e^{u \cdot \alpha_{\frac{k}{2}+j}} + e^{u \cdot \alpha_{\frac{k}{2}+2n-j}} \right) + e^{u \cdot \alpha_{\frac{k}{2}+n}} - 2n.$$

Noting that $\alpha_{\frac{k}{2}+i} \cdot \alpha'_j = \alpha_{\frac{k}{2}+2n-i} \cdot \alpha'_j = \alpha'_i \cdot \alpha'_j$, folding gives

$$\begin{aligned} \Phi &= e^{\phi \cdot \alpha'_0} + 2 \sum_{j=1}^{n-1} e^{\phi \cdot \alpha'_j} + e^{\phi \cdot \alpha'_n} - 2n \\ &= \sum_{j=0}^n n_j \left(e^{\alpha'_j \cdot \phi} - 1 \right) \end{aligned} \tag{2.2}$$

with $\{n_i\}$ the marks of the $c_n^{(1)}$ algebra. Hence, for any choice of even k , this prescription sends $a_{2n-1}^{(1)}$ ATFT to $c_n^{(1)}$ ATFT.

2.1.2 Case 2: h even, k odd

This case is quite different, in as much as all nodes are identified with one other. This case gives $a_{2n-1}^{(1)} \rightarrow d_n^{(2)}$ (despite the notation $d_n^{(2)}$ has just n nodes instead of $n+1$). There are

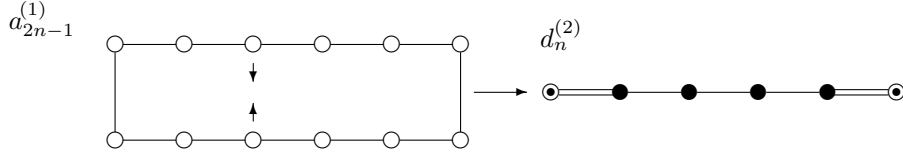


Figure 2: $a_{2n-1}^{(1)} \rightarrow d_n^{(2)}$.

(i, j)	$\alpha'_i \cdot \alpha'_j$
$(0, 0)$	$\frac{1}{2}$
$(s, s) \ s \neq 0, n$	1
(n, n)	$\frac{1}{2}$
$(s, s+1)$	$-\frac{1}{2}$

Table 2: Inner products of $d_2^{(1)}$ root space.

equivalent choices of α'_0 since the Dynkin diagram looks the same from both ends. The choice $\alpha'_0 = \frac{\alpha_{\frac{k+1}{2}} + \alpha_{\frac{k-1}{2} + 2n}}{2}$ is made, which includes α_0 when $k = \pm 1$. This folding is shown diagrammatically in figure 2; note that the black-in-white nodes have length $\frac{1}{\sqrt{2}}$.

The identification of roots in the folded theory is chosen to be

$$\alpha'_i = \frac{\alpha_{\frac{k+1}{2}+i} + \alpha_{\frac{k-1}{2}+2n-i}}{2}$$

and so

$$u = \sum_{j=0}^{2n-1} u_j \alpha_j \longrightarrow \phi = \sum_{j=0}^{n-1} \phi_j \frac{\alpha_{\frac{k+1}{2}+j} + \alpha_{\frac{k-1}{2}-j}}{2} = \sum_{j=0}^{n-1} \phi_j \alpha'_j$$

The normalisation of the $d_n^{(2)}$ roots here is non-canonical so this could be viewed with suspicion, however the inner products in table 2 are precisely half of the standard inner products of $d_n^{(2)}$. The remedy, if conventional normalisation is required, is simply to rescale the roots $\alpha'_i \rightarrow \sqrt{2}\alpha'_i$.

The potential obtained by this folding is

$$\Phi = 2 \sum_{j=0}^{n-1} n_j \left(e^{\alpha'_j \cdot \phi} - 1 \right) \quad (2.3)$$

since all the marks are equal to one in $d_n^{(2)}$. This potential obtained is twice the standard $d_n^{(2)}$ potential. The factor of two may be removed in the action by an isotropic space-time rescaling: $x \rightarrow \frac{x}{\sqrt{2}}, t \rightarrow \frac{t}{\sqrt{2}}$.

2.1.3 Case 3: h odd

This case, illustrated by figure 3, covers $a_{2n}^{(1)} \rightarrow a_{2n}^{(2)}$, so $h = 2n + 1$. Since the indices on the roots of $a_{2n}^{(1)}$ are identified modulo this odd Coxeter number, there is no longer a notion of odd

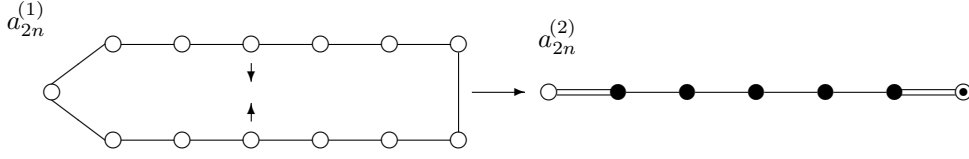


Figure 3: $a_{2n}^{(1)} \rightarrow a_{2n}^{(2)}$.

(i, j)	$\alpha'_i \cdot \alpha'_j$
$(0, 0)$	2
$(p, p) \ p \neq 0, n$	1
(n, n)	$\frac{1}{2}$
$(p, p+1) \ p \neq 0$	$-\frac{1}{2}$
$(0, 1)$	-1

Table 3: Inner products of $a_{2n}^{(2)}$ root space.

and even indices (e.g., $\alpha_1 \equiv \alpha_{2n+2}$). Due to this, k odd and k even give the same folding, and so the choice will be made that k is even (to ensure that all foldings are considered, a suitable range is $k = 0, 2, 4, \dots, 4n$).

This case contains one self-identified node, $\alpha_{\frac{k}{2}}$ which will be chosen to correspond to α'_0 . There are $n+1$ roots in the folded theory and the identification made is

$$\alpha'_i = \frac{\alpha_{\frac{k}{2}+i} + \alpha_{\frac{k}{2}+2n+1-i}}{2}.$$

The inner products are given in table 3.

The potential becomes

$$\Phi = \sum_{j=0}^n n_j \left(e^{\alpha'_j \cdot \phi} - 1 \right) \quad (2.4)$$

with $n_0 = 1$ and $n_i = 2$ for $i \neq 0$. Exactly as required for the folding to give $a_{2n}^{(2)}$ ATFT.

Thus it is shown that these foldings are valid, as expected from [9]. Given this, it is reasonable to consider the construction of solitons for $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ using these foldings (as this is the only folding process considered in this paper, these ATFTs will be collectively referred to as ‘the folded theories’).

3 Solitons and folding

This section details the construction of solitons in the folded theories, all of which are $a_r^{(1)}$ solitons possessing a certain symmetry (namely $\tau_i = \tau_{k+h-i}$). In this context, $a_r^{(1)}$ solitons must be introduced first.

3.1 $a_r^{(1)}$ solitons

Hirota methods [12] are used to find soliton solutions in the ATFTs. For this paper the starting point will be $a_r^{(1)}$, as it is the theories obtained from folding this that are of interest. For a general ATFT the soliton solutions may be written in the form [5]

$$u = -\frac{1}{\beta} \sum_{j=0}^r \eta_j \alpha_j \ln \tau_j \quad (3.1)$$

where $\eta_i = \frac{2}{\alpha_i \cdot \alpha_i}$ with no sum implied and the ATFT corresponds to an affine Dynkin diagram with $r+1$ nodes.

The tau functions $\{\tau_j\}$ depend only upon the root data (i.e., which ATFT is being considered) and do not depend upon the coupling. In order to find the tau functions (3.1) is used in the equation of motion (1.3) along with a decoupling [4]. In general the equation to be solved is

$$\eta_i (\ddot{\tau}_i \tau_i - \dot{\tau}_i^2 - \tau_i'' \tau_i + \tau_i'^2) - n_i \left(\prod_{j=0}^r \tau_j^{-\eta_j \alpha_j \cdot \alpha_i} - 1 \right) \tau_i^2 = 0. \quad (3.2)$$

Note that the kinetic terms are always in Hirota bilinear form $\ddot{\tau}_i \tau_i - \dot{\tau}_i^2 - \tau_i'' \tau_i + \tau_i'^2 = \frac{1}{2} (D_t^2 - D_x^2) \tau_i \cdot \tau_i$, but the potential is only in true bilinear form for $a_r^{(1)}$. For $a_r^{(1)}$, $\eta_i = n_i = 1$ for all i , so (3.1) becomes

$$u = -\frac{1}{\beta} \sum_{j=0}^r \alpha_j \ln \tau_j \quad (3.3)$$

while the equation the tau functions must obey simplifies greatly to

$$\ddot{\tau}_i \tau_i - \dot{\tau}_i^2 - \tau_i'' \tau_i + \tau_i'^2 = \tau_{i-1} \tau_{i+1} - \tau_i^2. \quad (3.4)$$

3.2 Folding $a_r^{(1)}$ solitons

In all cases folding is achieved by identifying α_i with α_{k+h-i} , so to fold the $a_r^{(1)}$ ATFT set $u_i = u_{k+h-i}$. In the soliton Ansatz (3.3) this is tantamount to having $\tau_i = \tau_{k+h-i}$. Any $a_r^{(1)}$ soliton with this property is also a soliton of the folded theory. A summary of the tau function identifications required for the folded solitons to fit the Ansatz (3.1) is given in table 4.

The possibility of obtaining solitons in the folded theories is dependent on the relation $\tau_i = \tau_{k+h-i}$ having solutions for all i , so the tau functions will need to be examined. It is already known that for $c_n^{(1)}$ in the case $k=0$ that such a relation is possible [5].

3.3 Tau functions

3.3.1 One soliton solution of $a_r^{(1)}$

The tau functions for the one soliton solution (of species p) of $a_r^{(1)}$ are of the form

$$\tau_j = 1 + \omega^{pj} E_p. \quad (3.5)$$

	$c_n^{(1)}$	$d_n^{(2)}$	$a_{2n}^{(2)}$
τ'_0	$\tau_{\frac{k}{2}}$	$(\tau_{\frac{k+1}{2}})^{\frac{1}{2}}$	$\tau_{\frac{k}{2}}$
$\tau'_i \ i \neq 0, n-1, n$	$\tau_{\frac{k}{2}+i}$	$\tau_{\frac{k+1}{2}+i}$	$\tau_{\frac{k}{2}+i}$
τ'_{n-1}	$\tau_{\frac{k}{2}+n-1}$	$(\tau_{\frac{k+1}{2}+n-1})^{\frac{1}{2}}$	$\tau_{\frac{k}{2}+n-1}$
τ'_n	$\tau_{\frac{k}{2}+n}$	—	$(\tau_{\frac{k}{2}+n})^{\frac{1}{2}}$

Table 4: Tau function identifications in the folded theories.

In (3.5), $\omega = e^{\frac{2\pi i}{r+1}}$, and $p = 1, \dots, r$ (note that $p = 0$ just gives the trivial solution), so that ω^p encompasses the $(r+1)$ -th roots of unity (where $r+1 = h$, the Coxeter number). The expression E_p is given by $E_p = e^{a_p x - b_p t + c_p}$, c_p is constant: its imaginary component determines the topological charge of the soliton under consideration- there are up to $r+1$ sectors of differing topological charge [15, 6]. In order to avoid singularities in the soliton solution (avoiding $\tau_j = 0$) this imaginary part of c_p should be chosen carefully. The real part of c_p gives the position of the centre of mass of the soliton at time $t = 0$ and may be chosen arbitrarily. $a_p = \lambda_p \cosh \theta$ and $b_p = \lambda_p \sinh \theta$ with $\mathcal{R}(\theta) > 0$ for a right-moving soliton (θ is the rapidity- or, at least, $\mathcal{R}(\theta)$ is). λ_p is taken to be the positive square root of the p th eigenvalue of the matrix $(NC)_{ij} = n_i \alpha_i \cdot \alpha_j$. For $a_r^{(1)}$ these are merely the ordered eigenvalues of the Cartan matrix of the (affine) Lie algebra, given by

$$\lambda_p^2 = 4 \sin^2 \left(\frac{\pi p}{r+1} \right) \quad (3.6)$$

with most eigenvalues appearing twice. $p = 0$, $\lambda_0^2 = 0$ (in all cases), and $p = n$, $\lambda_n^2 = 4$, (in the case $r = 2n - 1$) are the only eigenvalues which aren't degenerate. It should be noted that the square masses of the $a_r^{(1)}$ solitons are proportional to these eigenvalues, so it is not erroneous to refer to $\{\lambda_p\}$ as the 'masses' of the solitons. One thing to note is that the $\{\lambda_p\}$ in the folded theories obtained from $a_r^{(1)}$ in this paper are exactly the same, only that the degeneracy has been removed (the 'masses' of the basic soliton solutions in the folded theories are typically $\{2\lambda_p\}$). This ought to be contrasted to the cases of $d_n^{(2)}$ and $a_{2n}^{(2)}$ when folded from $d_s^{(1)}$ theories- there the eigenvalues of NC for $d_s^{(1)}$ need to be rescaled to obtain those of the folded theories [6].

Note at this point that there is but one non-trivial soliton of $a_r^{(1)}$ which is compatible with folding (i.e., has $\tau_j = \tau_{h+k-j}$), which is the n soliton of $a_{2n-1}^{(1)}$ with k even. In that case $\tau_j = 1 + (-1)^j E = \tau_{2n+k-j}$, since $(-1)^{2n+k} = 1$. This soliton is the species n single soliton of $c_n^{(1)}$, as well as being a *bona fide* a_1 soliton.

3.3.2 Single solitons in the folded theories

The tau functions of a two soliton solution (one of type p ; the other of type q) in $a_r^{(1)}$ have the form

$$\tau_j = 1 + \omega^{pj} E_p + \omega^{qj} E_q + A^{(pq)} \omega^{(p+q)j} E_p E_q \quad (3.7)$$

where there is now an interaction parameter

$$A^{(pq)} = -\frac{(a_p - a_q)^2 - (b_p - b_q)^2 - \lambda_{p-q}^2}{(a_p + a_q)^2 - (b_p + b_q)^2 - \lambda_{p+q}^2} \quad (3.8)$$

and $a_p = \lambda_p \cosh \theta_1$, $a_q = \lambda_q \cosh \theta_2$, etc. (i.e., there are two rapidities to be considered). If p and q are the same, and both rapidities are the same, then (3.8) vanishes, and in fact (3.7) describes a one soliton solution (from this point of view the n soliton of $c_n^{(1)}$ may also be thought of as a two soliton solution of $a_{2n-1}^{(1)}$).

For $a_r^{(1)}$ solitons to also be solitons of the folded theories it must be the case that $\tau_j = \tau_{k+h-j}$. After folding (3.7) is to be interpreted as a one soliton solution, so the constituent solitons must be given the same centre of mass, $\mathcal{R}(c_p) = \mathcal{R}(c_q)$, as well as the same rapidity, $\theta_1 = \theta_2$. Also, there is little chance of satisfying $\tau_j = \tau_{k+h-j}$ if the quadratic term in E depends on j : the conclusion is that $\omega^{p+q} = 1$ and so $q = h - p$. Consequently, the solitons being paired up possess the same mass, meaning that $a_p = a_q$, $b_p = b_q$. Thus, E_p and E_q may only differ in $\mathcal{I}(c)$.

From (3.8), with $\theta_1 = \theta_2$,

$$A^{(p(h-p))} = -\frac{-\lambda_{2p}^2}{4a_p^2 - 4b_p^2} = \frac{4 \sin^2 \left(\frac{2\pi p}{r+1} \right)}{16 \sin^2 \left(\frac{\pi p}{r+1} \right)} = \cos^2 \left(\frac{\pi p}{r+1} \right) \equiv A. \quad (3.9)$$

Thus, the tau functions compatible with folding (to a one soliton folded solution) all possess the form

$$\tau_j = 1 + \left(\omega^{pj} + \omega^{pk-pj} \right) E_p + A \omega^{pk} E_p^2 \quad (3.10)$$

which in $a_r^{(1)}$ naïvely appears to have the interpretation of a species p $a_r^{(1)}$ soliton joined to a species $h - p$ soliton which has its topological charge shifted into a different sector. It turns out however, that the topological charges of the folded solitons only depend on whether k is odd or even and so all possibilities are actually encompassed by $k = 0$ and $k = 1$. Nonetheless, this paper will continue to work with arbitrary k .

It can be shown that these folded solitons are the same as those found in [6] with the identifications in table 4.

The relative simplicity of $a_r^{(1)}$ allows easier construction of the multisoliton solutions than directly in the folded models, or via $d_s^{(1)}$ which would otherwise be necessary for $d_n^{(2)}$ and $a_{2n}^{(2)}$. Thus, some results are given in the next section which have not previously been published (the process was considered before but not included in any publication³).

3.3.3 Other solutions

Once the basic soliton solutions of the folded models have been found, multisolitons in these models can be constructed; however, this requires knowledge of the generally complicated interaction parameters of the folded model. This problem may be obviated by instead constructing

³G. M. T. Watts, private communication.

these multisolitons in the $a_r^{(1)}$ model. This was noted in [16], but was only applied to $c_n^{(1)}$. A formula for multisoliton solutions in $a_r^{(1)}$ is [4]

$$\tau_j = \sum_{\mu_1=0}^1 \dots \sum_{\mu_N=0}^1 \exp \left[\sum_{i=1}^N \mu_i \ln (\omega^{p_i j} E_{p_i}) + \sum_{1 \leq i < j \leq N} \mu_i \mu_j \ln (A^{(p_i p_j)}) \right]. \quad (3.11)$$

Any solution to (3.11) with $\tau_j = \tau_{h+k-j}$ will describe a soliton configuration in the folded theory. In particular, the two soliton solution takes the form

$$\begin{aligned} \tau_j = 1 &+ \left(\omega^{pj} + \omega^{p(k-j)} \right) E_p + \left(\omega^{qj} + \omega^{q(k-j)} \right) E_q + A^{(12)} \omega^{pk} E_p^2 + A^{(34)} \omega^{qk} E_q^2 \\ &+ A^{(13)} \left(\omega^{pj+qj} + \omega^{p(k-j)+q(k-j)} \right) E_p E_q + A^{(14)} \left(\omega^{pj+q(k-j)} + \omega^{p(k-j)+qj} \right) E_p E_q \\ &+ A^{(12)} A^{(13)} A^{(14)} \omega^{pk} \left(\omega^{qj} + \omega^{q(k-j)} \right) E_p^2 E_q \\ &+ A^{(34)} A^{(13)} A^{(14)} \omega^{qk} \left(\omega^{pj} + \omega^{p(k-j)} \right) E_p E_q^2 \\ &+ A^{(12)} A^{(34)} \left(A^{(13)} \right)^2 \left(A^{(14)} \right)^2 \omega^{pk+qk} E_p^2 E_q^2. \end{aligned} \quad (3.12)$$

Note that there is no need to convert these tau functions to those that are conventional from (3.1) as these tau functions with the Ansatz (3.3) and appropriate identification of roots are the solitons of the folded theories. Note that (3.12) contains four interaction parameters- a fact that is not obvious, should one wish to construct folded solitons using the folded theory as a starting point.

Using that $a_p = \lambda_p \cosh \theta_1, a_q = \lambda_q \cosh \theta_2, b_p = \lambda_p \sinh \theta_1, b_q = \lambda_q \sinh \theta_2$ and denoting the two rapidities by $\theta_1 = \theta + \psi$ and $\theta_2 = \theta - \psi$ gives the interaction parameters as

$$A^{(13)} = \frac{\lambda_{p-q}^2 + (\lambda_p + \lambda_q)^2 \sinh^2 \psi - (\lambda_p - \lambda_q)^2 \cosh^2 \psi}{(\lambda_p + \lambda_q)^2 \cosh^2 \psi - (\lambda_p - \lambda_q)^2 \sinh^2 \psi - \lambda_{p+q}^2} \quad (3.13)$$

$$A^{(14)} = \frac{\lambda_{p+q}^2 + (\lambda_p + \lambda_q)^2 \sinh^2 \psi - (\lambda_p - \lambda_q)^2 \cosh^2 \psi}{(\lambda_p + \lambda_q)^2 \cosh^2 \psi - (\lambda_p - \lambda_q)^2 \sinh^2 \psi - \lambda_{p-q}^2} \quad (3.14)$$

$$A^{(12)} = \cos^2 \left(\frac{\pi p}{r+1} \right) \quad (3.15)$$

$$A^{(34)} = \cos^2 \left(\frac{\pi q}{r+1} \right). \quad (3.16)$$

Among the two soliton solutions there are two interesting cases that can occur when the the relative rapidity $\theta_1 - \theta_2 = 2\psi$ between the solitons is imaginary:

- The solitons possess fusing rules, which are just $a_r^{(1)}$ fusing rules. Fusion of the solitons occurs when the denominator of $A^{(13)}$ in equation (3.13) vanishes (one should first make the redefinitions $E_p \rightarrow (A^{(13)})^{-\frac{1}{2}} E_p$ and $E_q \rightarrow (A^{(13)})^{-\frac{1}{2}} E_q$ in equation (3.12)). This occurs when $\psi = \pm i \frac{\pi(p+q)}{2(r+1)} \equiv \pm i \tilde{\psi}$. The resulting tau functions describe a species $s = p+q$ single folded soliton with rapidity $\tilde{\theta} = \theta + i \frac{\pi(p-q)}{2(r+1)}$.
- The existence of breather solutions in ATFT has been known for some time [17] and solutions have been considered in Hirota form for $a_r^{(1)}$ [19] and $d_4^{(1)}$ [20]. For equation

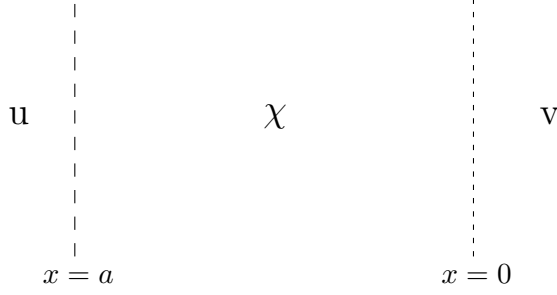


Figure 4: Two unfused defects in $a_r^{(1)}$.

(3.12) to describe a folded breather the constituent solitons must be of the same species, $p = q$, with the same centre of mass $\mathcal{R}(c_1) = \mathcal{R}(c_2)$ (where $E_p \rightarrow E_1$ and $E_q \rightarrow E_2$) and with imaginary rapidity difference, $\psi = i\tilde{\psi} - 2\tilde{\psi}$ must be less than the fusing angle. Note that in the cases of $d_n^{(2)}$ and $a_{2n}^{(2)}$ these breather tau functions are also the tau functions of particular breathers in $d_s^{(1)}$.

4 Fusing defects

This section deals with ‘fusing’ defects, i.e., combining two defects (both of type I in the language of [3]) to form a single (type II) defect. There are two kinds of (type I) integrable defect known in $a_r^{(1)}$ [2] which are referred to in the introduction as B and B^T defects. One kind of defect may be transformed into the other kind by considering the roots of the algebra $\{\alpha_i\}$ in a reversed order- this is reminiscent of the folding procedure which identifies u_i with u_{k+h-i} .

4.1 Two defect system and time delays

The starting point here will be consideration of $a_r^{(1)}$ with two unfused defects: a B defect at $x = a < 0$ and a B^T defect at $x = 0$. This is illustrated in figure 4. The Lagrange density describing this system may be written as

$$\begin{aligned} \mathcal{L} = & \theta(a-x)\mathcal{L}_u + \delta(a-x) \left(\frac{1}{2}uA\dot{u} + uB\dot{\chi} + \frac{1}{2}\chi A\dot{\chi} - D^{(1)}(u, \chi) \right) \\ & + \theta(x-a)\theta(-x)\mathcal{L}_\chi + \delta(x) \left(-\frac{1}{2}\chi A\dot{\chi} + \chi B^T\dot{v} - \frac{1}{2}vA\dot{v} - D^{(2)}(\chi, v) \right) + \theta(x)\mathcal{L}_v \end{aligned} \quad (4.1)$$

where $A = 1 - B$ and $B + B^T = 2$ with \mathcal{L}_u being a Lagrange density in the form of (1.1); \mathcal{L}_v and \mathcal{L}_χ similarly. In section 2 it was argued that B may be written

$$B = 2 \sum_{j=0}^r (\lambda_j - \lambda_{j+1}) \lambda_j^T \quad (4.2)$$

where $\lambda_q = 0$ for some q (depending on how the fundamental weights are defined) and B^T is simply the transpose of B .

The defect potentials are given by

$$D^{(1)} = d \sum_{j=0}^r e^{\frac{1}{2}\alpha_j(B^T u + B\chi)} + \frac{1}{\tilde{d}} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j B(u-\chi)} \quad (4.3)$$

$$D^{(2)} = \tilde{d} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j(B^T v + B\chi)} + \frac{1}{d} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j B(v-\chi)} \quad (4.4)$$

where d and \tilde{d} are free parameters, referred to here as the defect parameters- they may be interpreted as relating to the rapidities of solitons when Λ_p and $\tilde{\Lambda}_p$ have poles/zeros.

In addition to the bulk affine Toda equations of motion, the Euler-Lagrange equations of (4.1) give the defect conditions

$$u' = A\dot{u} + B\dot{\chi} - D_u \big|_{x=a} \quad (4.5)$$

$$\chi' = -A\dot{\chi} + B^T \dot{u} + D_\chi^{(1)} \big|_{x=a} \quad (4.6)$$

$$\chi' = -A\dot{\chi} + B^T \dot{v} - D_\chi^{(2)} \big|_{x=0} \quad (4.7)$$

$$v' = A\dot{v} + B\dot{\chi} + D_v \big|_{x=0} . \quad (4.8)$$

In (4.5), (4.8) and subsequently, where D is not given a superscript it will be taken to mean $D = D^{(1)} + D^{(2)}$.

Consider evolving a right-moving species p soliton through this system. An Ansatz is made as in [2] that the soliton should retain its form, only picking up a time delay and phase shift from the defect, so consideration of the defect on the left means that

$$u = - \sum_{j=0}^r \alpha_j \ln \tau_j^u$$

$$\chi = - \sum_{j=0}^r \alpha_j \ln \tau_j^\chi$$

where the tau functions have the form of (3.5)

$$\tau_j^u(a) = 1 + \omega^{pj} E_p(a)$$

$$\tau_j^\chi(a) = 1 + \omega^{pj} \Lambda_p E_p(a) .$$

By considering (4.5) and (4.6) (take the inner product of each with α_i) it is found, as in [2] that Λ_p is given by (1.9). The time delay to the soliton from this defect is then given by $\Delta t = \frac{1}{b_p} \ln |\Lambda_p|$ while the phase of Λ_p may be absorbed into $\mathcal{I}(c_p)$ (possibly affecting the topological charge). Note the possibility of a pole in (1.9) hints at the possibility of solitons being absorbed and emitted- this change in soliton number is not surprising as the $x = a$ defect conditions give a Bäcklund transformation for $a_r^{(1)}$ (the $x = 0$ defect conditions give another Bäcklund transformation).

The other defect at $x = 0$ produces a similar ‘delay’

$$\tau_j^\chi(0) = 1 + \omega^{pj} \Lambda_p E_p(0)$$

$$\tau_j^v(0) = 1 + \omega^{pj} \Lambda_p \tilde{\Lambda}_p E_p(0)$$

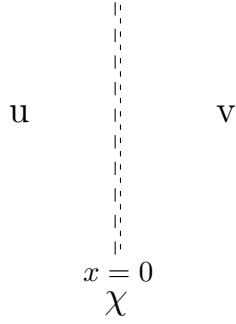


Figure 5: The fused $a_r^{(1)}$ defect.

where $\tilde{\Lambda}_p$ is given by (1.10). The field on the right, v then differs from what it would in the no defect case by the factor $\Lambda_p \tilde{\Lambda}_p$.

The general two soliton solution of $a_r^{(1)}$ has tau functions given by (3.7). The constituent solitons (species p and species q) are delayed by the defects independently so the effect of the defects is

$$\begin{aligned}\tau_j^u &= 1 + \omega^{pj} E_p + \omega^{qj} E_q + A^{(pq)} \omega^{(p+q)j} E_p E_q \\ \tau_j^\chi &= 1 + \omega^{pj} \Lambda_p E_p + \omega^{qj} \Lambda_q E_q + A^{(pq)} \omega^{(p+q)j} \Lambda_p \Lambda_q E_p E_q \\ \tau_j^v &= 1 + \omega^{pj} \Lambda_p \tilde{\Lambda}_p E_p + \omega^{qj} \Lambda_q \tilde{\Lambda}_q E_q + A^{(pq)} \omega^{(p+q)j} \Lambda_p \Lambda_q \tilde{\Lambda}_p \tilde{\Lambda}_q E_p E_q .\end{aligned}\quad (4.9)$$

This has been verified the lowest non-trivial order (i.e., consideration of (4.5)-(4.8) at order $E_p E_q$). Its veracity may also be argued in terms of the commutability of Bäcklund transformations: evolving the species p and species q solitons through a defect and then combining into a two soliton solution ought to give the same as combining into a two soliton solution and then evolving through the defect.

It turns out that (4.9) retains its relevance when the defects are fused and folded, this is argued in the following sections.

4.2 A fused $a_r^{(1)}$ defect and energy-momentum conservation

Fusing is done at the Lagrangian level by taking $a \rightarrow 0$ in (4.1), leaving

$$\begin{aligned}\mathcal{L} &= \theta(-x) \mathcal{L}_u + \theta(x) \mathcal{L}_v \\ &+ \delta(x) \left(\frac{1}{2} u A \dot{u} + u B \dot{\chi} - v B \dot{\chi} - \frac{1}{2} v A \dot{v} - D^{(1)} - D^{(2)} \right) .\end{aligned}\quad (4.10)$$

There no longer exists any bulk for the field χ ; it is effectively trapped in the defect (as illustrated in figure 5) and hence may be referred to as an ‘auxiliary field’. Note also that (4.10) bears some similarity to (1.11) though there is no direct coupling of the bulk fields u and v -this is to be expected here as the fused defect arises from two separated defects and locality dictates that u could only couple to v via the field χ . In fact, [21] and [23] use a modified type II Ansatz which eliminates the direct coupling of the bulk fields found in [3].

The Euler Lagrange equations of (4.10) give now the bulk equations for u and v as well as

the defect conditions at $x = 0$

$$u' = A\dot{u} + B\dot{\chi} - D_u \quad (4.11)$$

$$v' = A\dot{v} + B\dot{\chi} + D_v \quad (4.12)$$

$$B^T \dot{u} + D_\chi^{(1)} = B^T \dot{v} - D_\chi^{(2)} . \quad (4.13)$$

There are now three vector equations instead of the four of (4.5)-(4.8) and it is seen that (4.13) also arises from identification (4.6) and (4.7) when $a = 0$. Thus it follows that any solution to (4.5)-(4.8) with $a = 0$ must also solve (4.11), (4.12) and (4.13) and so the time delays previously found and (4.9) remain valid for the fused defect.

Conservation of a modified energy and momentum for the fused defect is now shown- this strongly indicates classical integrability of the fused defect, but is mainly done in order to set the scene for energy-momentum conservation of the folded defects.

4.2.1 Energy conservation

Consider the bulk contribution to the energy, i.e., the integral of T^{00} of the stress tensor which derives from (4.10).

$$E = \int_{-\infty}^0 \frac{1}{2} \dot{u} \cdot \dot{u} + \frac{1}{2} u' \cdot u' + U \, dx + \int_0^\infty \frac{1}{2} \dot{v} \cdot \dot{v} + \frac{1}{2} v' \cdot v' + V \, dx$$

where U and V are the bulk potentials for the fields u and v respectively (the field χ , before fusing, would have a potential denoted X). Taking the time derivative of this and using the bulk equations of motion; $\ddot{u} - u'' = -U_u$, $\ddot{v} - v'' = -V_v$; gives

$$\dot{E} = \dot{u} \cdot u' - \dot{v} \cdot v' \big|_{x=0}$$

provided that the fields are constant at spatial infinity. This is now amenable to the use of the defect conditions (4.11), (4.12) and (4.13)

$$\begin{aligned} \dot{E} &= \dot{u} \cdot u' - \dot{v} \cdot v' \big|_{x=0} \\ &= \dot{u} (A\dot{u} + B\dot{\chi} - D_u) - \dot{v} (A\dot{v} + B\dot{\chi} + D_v) \\ &= \dot{\chi} (B^T \dot{u} - B^T \dot{v}) - \dot{u} \cdot D_u - \dot{v} \cdot D_v \\ &= -\dot{u} \cdot D_u - \dot{v} \cdot D_v - \dot{\chi} \cdot D_\chi = -\dot{D} . \end{aligned}$$

So, the quantity $E + D$, the bulk plus the defect energy, is conserved. In some sense this is trivial, as applying a Legendre transformation to the Lagrange density (4.10) and performing spatial integration gives the quantity $E + D$ as the Hamiltonian of the system.

It would appear that energy conservation makes no constraint on D , but there is one subtle implication in interpreting D as an energy contribution. The energy can be viewed as the sum of a positive and a negative helicity term, i.e., the sum of the lightcone momenta P^+ and P^- (with a possible factor depending on notation)- hence, the defect potential D (indeed $D^{(1)}$ and $D^{(2)}$ separately- from energy conservation of type I defects) may also be written as the sum of a positive and a negative helicity term. The form of $D^{(1)}$ and $D^{(2)}$ are already known and given by (4.3) and (4.4) and each is already written in two parts- the interpretation here is

that the terms with prefactor d and \tilde{d} have positive helicity while the d^{-1} and \tilde{d}^{-1} terms have negative helicity- the opposite labelling is equally possible. This interpretation of the helicities is a useful guide for the momentum conservation argument. Momentum is given by the difference of the lightcone momenta $P^+ - P^-$, so the defect contribution to momentum is expected to consist of the difference of a positive and a negative helicity term.

4.2.2 Momentum conservation

Here it is shown that the fused defect conserves momentum. Naïvely one might not expect any sort of momentum conservation from a defect ATFT given that placing the defect explicitly breaks spatial translation invariance. Nonetheless, [11] shows that not only do $a_r^{(1)}$ defects admit a modified conserved momentum, the conservation of energy and momentum alone is enough to specify the Lagrangian with the result being that the defect is integrable, given the Lax pair analysis of [2]. It is not guaranteed that momentum conservation in other types of defect implies integrability, but given the similarities of the different ATFTs, there is reason to believe that this is true.

The bulk contribution to the momentum is given by the integral of T^{01} , so

$$P = \int_{-\infty}^0 \dot{u} \cdot u' dx + \int_0^{\infty} \dot{v} \cdot v' dx .$$

Taking the time derivative and using the bulk equations of motion gives

$$\dot{P} = \frac{1}{2} \dot{u} \cdot \dot{u} + \frac{1}{2} u' \cdot u' - U - \frac{1}{2} \dot{v} \cdot \dot{v} - \frac{1}{2} v' \cdot v' + V|_{x=0} \quad (4.14)$$

so long as the fields and potentials are constant (in vacuum) at spatial infinity. The aim again is to show that there is a conserved quantity by showing that \dot{P} is a total time derivative of a quantity that exists at $x = 0$, so the first step is to remove the spatial derivatives via (4.11) and (4.12) to obtain

$$\begin{aligned} u' \cdot u' &= (\dot{u} A^T + \dot{\chi} B^T - D_u) \cdot (A \dot{u} + B \cdot \chi - D_u) \\ &= \dot{u} A^T A \dot{u} + \dot{\chi} B^T B \dot{\chi} + D_u^2 + 2 \dot{\chi} B^T A \dot{u} - 2 \dot{u} A^T D_u - 2 \dot{\chi} B^T D_u \end{aligned} \quad (4.15)$$

and similarly

$$\begin{aligned} v' \cdot v' &= (\dot{v} A^T + \dot{\chi} B^T + D_v) \cdot (A \dot{v} + B \cdot \chi + D_v) \\ &= \dot{v} A^T A \dot{v} + \dot{\chi} B^T B \dot{\chi} + D_v^2 + 2 \dot{\chi} B^T A \dot{v} + 2 \dot{v} A^T D_v + 2 \dot{\chi} B^T D_v . \end{aligned} \quad (4.16)$$

Noting that $A^T A = B^T B - 1 = B B^T - 1$, $B^T A = A B^T$, (4.15) and (4.16) give

$$\begin{aligned} u' \cdot u' - v' \cdot v' &= \dot{u} (B B^T - 1) \dot{u} - \dot{v} (B B^T - 1) \dot{v} + 2 \dot{\chi} A (B^T \dot{u} - B^T \dot{v}) + D_u^2 - D_v^2 \\ &\quad - 2 \dot{u} A^T D_u - 2 \dot{v} A^T D_v - 2 \dot{\chi} (B^T D_u + B^T D_v) \\ &= \dot{u} (B B^T - 1) \dot{u} - \dot{v} (B B^T - 1) \dot{v} + D_u^2 - D_v^2 \\ &\quad - 2 \dot{u} A^T D_u - 2 \dot{v} A^T D_v - 2 \dot{\chi} (B^T D_u + A D_{\chi}^{(1)} + B^T D_v + A D_{\chi}^{(2)}) \end{aligned} \quad (4.17)$$

where (4.13) has been used to substitute for $B^T \dot{u} - B^T \dot{v}$. It is clear that (4.17) contains the terms $-\dot{u} \cdot \dot{u} + \dot{v} \cdot \dot{v}$, cancelling terms in the momentum expression, but there are still terms

which are quadratic in time derivatives to deal with, i.e., $\dot{u}BB^T\dot{u} - \dot{v}BB^T\dot{v}$; these terms can be dealt with by squaring and equating both sides of (4.13). As a general principle, it is sensible to pair up $D^{(1)}$ with the field u and $D^{(2)}$ with the field v which is why (4.13) is presented in its given form. Squaring both sides of (4.13) gives

$$\dot{u}BB^T\dot{u} - \dot{v}BB^T\dot{v} = -2\dot{u}BD_\chi^{(1)} - 2\dot{v}BD_\chi^{(2)} - D_\chi^{(1)} \cdot D_\chi^{(1)} + D_\chi^{(2)} \cdot D_\chi^{(2)} \quad (4.18)$$

and consequently, (4.14) reduces to

$$\begin{aligned} \dot{P} &= \frac{1}{2} (\dot{u} \cdot \dot{u} - \dot{v} \cdot \dot{v} + u' \cdot u' - v' \cdot v') - U + V \\ &= -U + V + \frac{1}{2} (D_u^2 - D_\chi^{(1)2} + D_\chi^{(2)2} - D_v^2) \\ &\quad - \dot{u} (A^T D_u + BD_\chi^{(1)}) - \dot{v} (A^T D_v + BD_\chi^{(2)}) \\ &\quad - \dot{\chi} (B^T D_u + AD_\chi^{(1)} + B^T D_v + AD_\chi^{(2)}) . \end{aligned} \quad (4.19)$$

For (4.19) to be a total time derivative strong constraints must be placed on the form of D . Fortunately, the form of D here is already known from [2] and the fusing process so it becomes a matter of testing that D has the right properties.

With the identification of helicities of terms in D made in section 4.2.1, as well as D being the sum of $D^{(1)}$ and $D^{(2)}$, the defect potential, (4.3) plus (4.4), splits into four parts given by

$$\begin{aligned} D^{(1)+} &= d \sum_{j=0}^r e^{\frac{1}{2}\alpha_j(B^T u + B\chi)} \equiv \sum_{j=0}^r a_j \\ D^{(2)+} &= \tilde{d} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j(B\chi + B^T v)} \equiv \sum_{j=0}^r \tilde{a}_j \\ D^{(1)-} &= \frac{1}{d} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j B(u-\chi)} \equiv \sum_{j=0}^r b_j \\ D^{(2)-} &= \frac{1}{\tilde{d}} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j B^T(\chi-v)} = \frac{1}{\tilde{d}} \sum_{j=0}^r e^{\frac{1}{2}\alpha_j B(v-\chi)} \equiv \sum_{j=0}^r \tilde{b}_j . \end{aligned} \quad (4.20)$$

Note that $D^{(2)}$ has precisely the same form as $D^{(1)}$ only with v instead of u ; consequently any property of $D^{(2)}$ may be drawn on by analogy to the corresponding property of $D^{(1)}$. Note now that

$$\begin{aligned} D_u^{(1)+} &= \sum_{j=0}^r \frac{1}{2} B \alpha_j a_j & D_\chi^{(1)+} &= \sum_{j=0}^r \frac{1}{2} B^T \alpha_j a_j \\ D_u^{(1)-} &= \sum_{j=0}^r \frac{1}{2} B^T \alpha_j b_j & D_\chi^{(1)-} &= - \sum_{j=0}^r \frac{1}{2} B^T \alpha_j b_j \end{aligned}$$

with similar expressions involving $D^{(2)}$. This gives the relations

$$\begin{aligned} B^T D_u^+ &= BD_\chi^{(1)+} \\ D_u^- &= -D_\chi^{(1)-} \\ B^T D_v^+ &= BD_\chi^{(2)+} \\ D_v^- &= -D_\chi^{(2)-} \end{aligned} \quad (4.21)$$

and so, using (4.21), (4.19) reduces to

$$\begin{aligned} \dot{P} = & -U + V + \frac{1}{2} \left(D_u^2 - D_\chi^{(1)2} + D_\chi^{(2)2} - D_v^2 \right) \\ & - \dot{u} (D_u^+ - D_u^-) - \dot{v} (D_v^+ - D_v^-) - \dot{\chi} (D_\chi^+ - D_\chi^-) . \end{aligned}$$

So, in order for momentum conservation to work, D must be quadratically related to U and V such that $\frac{1}{2} \left(D_u^2 - D_\chi^{(1)2} + D_\chi^{(2)2} - D_v^2 \right) = U - V$.

Noting that

$$\alpha_i B^T \alpha_j = \begin{cases} 2 & \text{if } i = j, \\ -2 & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{cases} \quad \alpha_i B \alpha_j = \begin{cases} 2 & \text{if } i = j, \\ -2 & \text{if } i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

and using the relations (4.21) and the form of D in (4.20) it is seen that

$$\frac{1}{2} \left(D_u^2 - D_\chi^{(1)2} \right) = D_u^+ D_u^- - D_\chi^{(1)+} D_\chi^{(1)-} = \frac{1}{2} \sum_{i,j=0}^r \alpha_i B^T \alpha_j a_i b_j = \sum_i a_i b_i - \sum_i a_i b_{i+1} = U - X$$

where X is the would-be bulk potential for χ . A similar relation holds for $D^{(2)}$, implying that $D_\chi^{(2)2} - D_v^2 = X - V$ and so it is true that $\frac{1}{2} \left(D_u^2 - D_\chi^{(1)2} + D_\chi^{(2)2} - D_v^2 \right) = U - V$ which finally gives

$$\dot{P} = - \left(\dot{D}^+ - \dot{D}^- \right)$$

meaning that there is a conserved momentum given by $P + D^+ - D^-$. It is conjectured that for the folded case that the same quantity, with the bulk fields u and v replaced in D by their folded counterparts, is conserved, i.e., the aim is to take \dot{P} and show that it in fact is equal to $-\left(\dot{D}^+ - \dot{D}^- \right)$.

Note that an analogous analysis holds if in the first instance in (4.1) the B defect is taken to be to the right of the B^T defect. This perhaps should come as no surprise as one may appeal to commutability given that the defect conditions describe Bäcklund transformations.

5 The folded defect

This section discusses of the folding of the fused defect system and by consideration of soliton time delays then energy and momentum conservation it strongly suggests the existence of classically integrable $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ defects. The process and the outcome are perhaps the two most salient aspects of this paper.

5.1 Folding the Lagrangian

There is from the outset an ambiguity over what is meant by the folding of the fused defect system of section 4.2. One can consider folding at the Lagrangian level before finding the Euler-Lagrange equations, termed as *a priori* folding in this paper. Another way to view folding is to first find the Euler-Lagrange equations of the unfolded Lagrangian and then afterwards

fold these equations, termed as *a posteriori* folding. The equivalence of the two points of view is investigated in appendix A. In this section the folding being considered is *a priori* folding: folding the Lagrangian (4.10).

The aim is to have the folded system describe a defect in the folded theory. In the bulk that means that the fields u and v must be folded, but at the defect it is not clear what should be done with the auxiliary field χ . Consideration of the time delays of soliton solutions, section 5.3, suggests that the field χ should not be folded.

One simplification that occurs in folding the Lagrangian (4.10) is that the self-coupling kinetic terms at the defect vanish because $\alpha'_i B \alpha'_j = \alpha'_i \cdot \alpha'_j$. Hence, since $A = 1 - B$,

$$\phi A \dot{\phi} = \psi A \dot{\psi} = 0. \quad (5.1)$$

This means that the Lagrangian (4.10) folds to

$$\begin{aligned} \mathcal{L} = & \theta(-x) \mathcal{L}_\phi + \theta(x) \mathcal{L}_\psi \\ & + \delta(x) \left(\phi B \dot{\chi} - \psi B \dot{\chi} - D^{(1)}(\phi, \chi) - D^{(2)}(\chi, \psi) \right) \end{aligned} \quad (5.2)$$

which fits the modified type II framework of [21].

In vector form the Euler-Lagrange equations are thus

$$\phi' = {}_p B \dot{\chi} - D_\phi \quad (5.3)$$

$$\psi' = {}_p B \dot{\chi} + D_\psi \quad (5.4)$$

$$B^T \dot{\phi} + D_\chi^{(1)} = B^T \dot{\psi} - D_\chi^{(2)} \quad (5.5)$$

The subscript p found in (5.3) and (5.4) in front of $B \dot{\chi}$ denotes ‘projected’. What is meant by this is explained in appendix A. Also tackled in appendix A is the relation between the folded and unfolded gradients, i.e., comparison of D_u to D_ϕ , the results of this analysis are summarised in the next section.

5.2 Content of the defect equations

In appendix A the equations (5.3), (5.4) and (5.5) are examined in greater depth. Here a summary is given of the most salient features.

Universal expressions are found for D_ϕ and D_ψ which are

$$D_\phi^+ = \sum_{j=0}^{h-1} \frac{1}{4} (B \alpha_j + B^T \alpha_{k+h-j}) a_j \quad (5.6)$$

$$D_\phi^- = \sum_{j=0}^{h-1} \frac{1}{4} (B^T \alpha_j + B \alpha_{k+h-j}) b_j \quad (5.7)$$

$$D_\psi^+ = \sum_{j=0}^{h-1} \frac{1}{4} (B \alpha_j + B^T \alpha_{k+h-j}) \tilde{a}_j \quad (5.8)$$

$$D_\psi^- = \sum_{j=0}^{h-1} \frac{1}{4} (B^T \alpha_j + B \alpha_{k+h-j}) \tilde{b}_j \quad (5.9)$$

while the gradient of D with respect to χ is given by

$$D_{\chi}^{+} = \sum_j \frac{1}{2} B^T \alpha_j (a_j + \tilde{a}_j) \quad (5.10)$$

$$D_{\chi}^{-} = - \sum_j \frac{1}{2} B^T \alpha_j (b_j + \tilde{b}_j) . \quad (5.11)$$

Examination of the components of (5.5) gives the algebraic constraints

$$D_{\chi i} + D_{\chi k+h-1-i} = 0$$

which may be put in the form

$$a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i} - b_i + b_{k+h-i} - \tilde{b}_i + \tilde{b}_{k+h-i} = 0 . \quad (5.12)$$

5.3 Time delay of solitons through the defect

From the argument of *a posteriori* folding it is seen that any solution to (4.11), (4.12) and (4.13) which has the symmetry of a folded field, namely $u_i = u_{k+h-i}$, $v_i = v_{k+h-i}$, must then satisfy the *a posteriori* folded equations (A.1), (A.2) and (A.3). Consequently, the fused defect time delays of equation (4.9) should be re-examined in the case of a folded soliton on the left, i.e., $q = h - p$, $E_q = \omega^{pk} E_p$. This gives

$$\tau_j^{\phi} = \tau_j^u = 1 + \left(\omega^{pj} + \omega^{pk-pj} \right) E_p + A \omega^{kj} E_p^2 \quad (5.13)$$

$$\tau_j^{\chi} = 1 + \left(\omega^{pj} \Lambda_p + \omega^{pk-pj} \Lambda_{h-p} \right) E_p + A \omega^{kj} \Lambda_p \Lambda_{h-p} E_p^2 \quad (5.14)$$

$$\tau_j^{\psi} \stackrel{?}{=} \tau_j^v = 1 + \left(\omega^{pj} \Lambda_p \tilde{\Lambda}_p + \omega^{qj} \Lambda_{h-p} \tilde{\Lambda}_{h-p} \right) E_p + A \omega^{kj} \Lambda_p \Lambda_{h-p} \tilde{\Lambda}_p \tilde{\Lambda}_{h-p} E_p^2 \quad (5.15)$$

where $A = \cos^2 \left(\frac{\pi p}{r+1} \right)$ and the delay factors are given by

$$\begin{aligned} \Lambda_p &= \frac{ie^{\theta} + d\omega^{\frac{p}{2}}}{ie^{\theta} + d\omega^{-\frac{p}{2}}} & \Lambda_{h-p} &= \frac{ie^{\theta} - d\omega^{-\frac{p}{2}}}{ie^{\theta} - d\omega^{\frac{p}{2}}} \\ \tilde{\Lambda}_p &= \frac{ie^{\theta} - \tilde{d}\omega^{-\frac{p}{2}}}{ie^{\theta} - \tilde{d}\omega^{\frac{p}{2}}} & \tilde{\Lambda}_{h-p} &= \frac{ie^{\theta} + \tilde{d}\omega^{\frac{p}{2}}}{ie^{\theta} + \tilde{d}\omega^{-\frac{p}{2}}} . \end{aligned} \quad (5.16)$$

It is clear that (5.13) represents a folded soliton, this is the choice that is made. Equally clear is that $\tau_j^{\chi} \neq \tau_{k+h-j}^{\chi}$, since $\Lambda_p \neq \Lambda_{h-p}$ except for $p = n$ in $c_n^{(1)}$ - consequently it is not possible to fold the auxiliary field χ , despite the algebraic constraints (5.12) implying that χ may be thought of as having the same number of degrees of freedom as a folded field has.

What is desired is that the field on the right, v represents a folded soliton, i.e., $\tau_j^v = \tau_{k+h-j}^v$, so that v may be replaced by ψ and the *a posteriori* defect equations satisfied. For this to be true it must be the case that

$$\Lambda_p \tilde{\Lambda}_p = \Lambda_{h-p} \tilde{\Lambda}_{h-p}$$

in which case every Λ may be absorbed into the definition of E_p as a time delay and phase shift. Note that this condition is the same condition as having the $a_r^{(1)}$ single soliton species p and

species $h - p$ solutions receiving the same delay through the fused defect. Thus, the condition for the soliton on the right of the defect to be in the folded theory is

$$0 = \Lambda_p \tilde{\Lambda}_p - \Lambda_{h-p} \tilde{\Lambda}_{h-p} = \frac{1}{\text{denom.}} \left[e^{2\theta} (\omega^p - \omega^{-p}) (d^2 - \tilde{d}^2) \right]$$

where ‘denom.’ is the common denominator obtained by multiplying all of the denominators of (5.16) together. So the defect represented by the Lagrangian (5.2) is only likely to be integrable if

$$\tilde{d} = \pm d \quad (5.17)$$

as this is what is required for the soliton solution (5.15) to be compatible with folding. Therefore, there are two possibilities then that give folded solitons for ψ

- When $\tilde{d} = -d$ it is the case that $\Lambda_p \tilde{\Lambda}_p = \Lambda_{h-p} \tilde{\Lambda}_{h-p} = 1$. i.e., all of the solitons receive a trivial time delay. Indeed in this case if $\psi = \phi$ is imposed then the defect part of the Lagrangian (5.2) vanishes- so there is no defect there. The interpretation of this is that the second defect is the anti-defect of the first- fusing them causes annihilation.
- When $\tilde{d} = d$ there is a non-trivial time delay (different for each p) so this should represent a *bona fide* defect which does not destroy the form of the solitons (a strong constraint, suggesting that the defect is integrable). Note that if θ and d are real then the time-delay is real too- there is no change of topological charge.

5.3.1 Algebraic constraints with $\tilde{d} = \pm d$

The algebraic constraints from (5.12) have the undesirable property of mixing helicities for general d and \tilde{d} . As such, only if the positive and negative helicity terms in (5.12) separately vanish will there be no mixing of helicities. Before folding the terms in D are

$$\begin{aligned} a_i &= d e^{u_i - u_{i+1} + \chi_i - \chi_{i-1}} \\ \tilde{a}_i &= \tilde{d} e^{v_i - v_{i+1} + \chi_i - \chi_{i-1}} \\ b_i &= d^{-1} e^{u_i - u_{i-1} - \chi_i + \chi_{i-1}} \\ \tilde{b}_i &= \tilde{d}^{-1} e^{v_i - v_{i-1} - \chi_i + \chi_{i-1}} . \end{aligned}$$

If $\tilde{d} = \pm d$ then

$$a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i} = d \left(e^{u_i - u_{i+1} + \chi_i - \chi_{i-1}} - e^{u_{k+h-i} - u_{k+h+1-i} + \chi_{k+h-i} - \chi_{k+h-1-i}} \right. \\ \left. \pm e^{v_i - v_{i+1} + \chi_i - \chi_{i-1}} \mp e^{v_{k+h-i} - v_{k+h+1-i} + \chi_{k+h-i} - \chi_{k+h-1-i}} \right)$$

$$b_i - b_{k+h-i} + \tilde{b}_i - \tilde{b}_{k+h-i} = d^{-1} \left(e^{u_i - u_{i-1} - \chi_i + \chi_{i-1}} - e^{u_{k+h-i} - u_{k+h-1-i} - \chi_{k+h-i} + \chi_{k+h-1-i}} \right. \\ \left. \pm e^{v_i - v_{i-1} - \chi_i + \chi_{i-1}} \mp e^{v_{k+h-i} - v_{k+h-1-i} - \chi_{k+h-i} + \chi_{k+h-1-i}} \right)$$

where the \pm and \mp are strictly correlated- the top set of relations refer to $\tilde{d} = d$, while the bottom to $\tilde{d} = -d$.

Folding the defect field theory, in all cases considered involves identifying $u_i = u_{k+h-i}$ and $v_i = v_{k+h-i}$. Leaving the folded fields in terms of $\{u_i\}$ and $\{v_i\}$ then gives

$$a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i} = d \left(e^{u_i - u_{i+1} + \chi_i - \chi_{i-1}} - e^{u_i - u_{i-1} + \chi_{k+h-i} - \chi_{k+h-1-i}} \right. \\ \left. \pm e^{v_i - v_{i+1} + \chi_i - \chi_{i-1}} \mp e^{v_i - v_{i-1} + \chi_{k+h-i} - \chi_{k+h-1-i}} \right)$$

whilst it is seen that, for either identification, $\tilde{d} = \pm d$,

$$b_i - b_{k+h-i} + \tilde{b}_i - \tilde{b}_{k+h-i} = -d^{-2} e^{-\chi_i + \chi_{i-1} - \chi_{k+h-i} + \chi_{k+h-1-i}} (a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i})$$

and so the positive and negative helicity algebraic constraints are equivalent

$$a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i} = 0 \quad \Longleftrightarrow \quad b_i - b_{k+h-i} + \tilde{b}_i - \tilde{b}_{k+h-i} = 0 .$$

In light of (5.12) then, the algebraic constraints become

$$\begin{aligned} D_{\chi_i}^+ + D_{\chi_{k+h-1-i}}^+ &= 0 \\ D_{\chi_i}^- + D_{\chi_{k+h-1-i}}^- &= 0 \end{aligned}$$

resulting in

$$a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i} = 0 \tag{5.18}$$

$$b_i - b_{k+h-i} + \tilde{b}_i - \tilde{b}_{k+h-i} = 0 . \tag{5.19}$$

It is noteworthy that condition (5.17) links the soliton time delay argument to the momentum conservation one. The condition ensures that solitons retain their form in the folded defect model and also ensures that the algebraic constraints do not mix helicities- which the momentum conservation argument of section 5.5 relies upon. It is thus assumed for the purposes on showing energy and momentum conservation that (5.17) is true, $\tilde{d} = \pm d$.

5.4 Energy conservation

For both energy and momentum conservation, the *a priori* folded defect equations; (5.3), (5.4) and (5.5); will be used. The process here is analogous to that of the energy conservation of the fused defect in section 4.2.1. The bulk energy is then modified by the defect such that

$$\begin{aligned} \dot{E} &= \dot{\phi} \cdot \phi' - \dot{\psi} \cdot \psi' |_{x=0} \\ &= \dot{\phi}_p B \dot{\chi} - \dot{\phi} \cdot D_\phi - \dot{\psi}_p B \dot{\chi} - \dot{\psi} \cdot D_\psi \\ &= \dot{\chi} \left(B^T \dot{\phi} - B^T \dot{\psi} \right) - \dot{\phi} \cdot D_\phi - \dot{\psi} \cdot D_\psi \\ &= -\dot{\phi} \cdot D_\phi - \dot{\chi} \cdot D_\chi - \dot{\psi} \cdot D_\psi = -\dot{D} \end{aligned}$$

so once again the quantity $E + D$ is conserved. Note that it is safe to drop the p subscript above as $\dot{\phi}$ and $\dot{\psi}$ both lie in the folded root space.

5.5 Momentum conservation

The momentum conservation by itself is the most involved of the calculations presented in this paper, but much simplified by employment of the relations (5.6)-(5.11) of section 5.2 along with the helicity conserving algebraic constraints (5.18) and (5.19). The bulk momentum is modified such that

$$\dot{P} = \frac{1}{2} \left(\phi' \cdot \phi' + \dot{\phi} \cdot \dot{\phi} - \psi' \cdot \psi' - \dot{\psi} \cdot \dot{\psi} \right) - \Phi + \Psi \tag{5.20}$$

where Φ and Ψ are the folded bulk potentials.

The first step in showing this momentum is conserved is to use the squares of (5.3) and (5.4), giving

$$\begin{aligned}\phi' \cdot \phi' &= ({}_p\dot{\chi}B^T - D_\phi)({}_pB\dot{\chi} - D_\phi) \\ &= {}_p\dot{\chi}B^T{}_pB\dot{\chi} - 2\dot{\chi}B^TD_\phi + D_\phi^2\end{aligned}$$

and

$$\begin{aligned}\psi' \cdot \psi' &= ({}_p\dot{\chi}B^T + D_\psi)({}_pB\dot{\chi} + D_\psi) \\ &= {}_p\dot{\chi}B^T{}_pB\dot{\chi} + 2\dot{\chi}B^TD_\psi + D_\psi^2.\end{aligned}$$

In both cases it is not known what the first term, ${}_p\dot{\chi}B^T{}_pB\dot{\chi}$, is in general; nor indeed is it even clear what it means. However, this issue drops out of the momentum conservation argument since

$$\phi' \cdot \phi' - \psi' \cdot \psi' = -2\dot{\chi}(B^TD_\phi + B^TD_\psi) + D_\phi^2 - D_\psi^2. \quad (5.21)$$

At this stage progress can be made by anticipating the final answer to be $\dot{P} = -(\dot{D}^+ - \dot{D}^-)$ which requires a term $-\dot{\chi}(D_\chi^+ - D_\chi^-)$. It is evident from the defect conditions; (5.3), (5.4) and (5.5); that the only place where terms dependent on $\dot{\chi}$ can appear in (5.20) stems from $\phi' \cdot \phi' - \psi' \cdot \psi'$. The conclusion is then that

$$B^TD_\phi^+ + B^TD_\psi^+ = D_\chi^+ \quad (5.22)$$

$$B^TD_\phi^- + B^TD_\psi^- = -D_\chi^-. \quad (5.23)$$

If (5.22) and (5.23) are true then (5.5) may be rewritten, noting that B^T is invertible, as

$$\dot{\phi} + D_\phi^+ - D_\phi^- = \dot{\psi} - D_\psi^+ + D_\psi^- \quad (5.24)$$

Note that the left-hand side of (5.24) is not equal $(B^T)^{-1}$ times the left-hand side of (5.5)-some rearrangement is necessary. Squaring both sides of (5.24) gives

$$\dot{\phi} \cdot \dot{\phi} - \dot{\psi} \cdot \dot{\psi} = -2\dot{\phi}(D_\phi^+ - D_\phi^-) - 2\dot{\psi}(D_\psi^+ - D_\psi^-) - (D_\phi^+ - D_\phi^-)^2 + (D_\psi^+ - D_\psi^-)^2 \quad (5.25)$$

and hence (5.21) and (5.25) may be combined to give

$$\phi' \cdot \phi' + \dot{\phi} \cdot \dot{\phi} - \psi' \cdot \psi' - \dot{\psi} \cdot \dot{\psi} = -2(\dot{D}^+ - \dot{D}^-) + 4D_\phi^+D_\phi^- - 4D_\psi^+D_\psi^-$$

to wit, (5.20) reduces to

$$\dot{P} = -\dot{D}^+ + \dot{D}^- + 2D_\phi^+D_\phi^- - 2D_\psi^+D_\psi^- - \Phi + \Psi. \quad (5.26)$$

Therefore, a modified momentum of $P + D^+ - D^-$ is conserved, provided

$$2D_\phi^+D_\phi^- - 2D_\psi^+D_\psi^- = \Phi - \Psi. \quad (5.27)$$

This argument would prove momentum conservation but there are three equations which must be proven which rely upon the particular form of D , i.e., (5.22), (5.23) and (5.27).

5.5.1 Proof of the relations (5.22) and (5.23)

In order to prove (5.22) and (5.23), all of the relations (5.6)-(5.11) will be required, along with the algebraic constraints (5.18) and (5.19). The first step is to note (5.10), given by

$$D_\chi^+ = \sum_i \frac{1}{2} B^T \alpha_j (a_j + \tilde{a}_j)$$

and to put the left-hand side of (5.22) into this form. Thus,

$$\begin{aligned} B^T D_\phi^+ + B^T D_\phi^+ &= \sum_j \frac{1}{4} B^T (B a_j + B \tilde{a}_j + B^T a_{k+h-j} + B^T \tilde{a}_{k+h-j}) \alpha_j \\ &= \sum_i \frac{1}{4} B^T ((B + B^T) a_j + (B + B^T) \tilde{a}_j + B^T (a_{k+h-j} - a_j) + B^T (\tilde{a}_{k+h-j} - \tilde{a}_j)) \alpha_j \\ &= \sum_j \frac{1}{2} B^T \alpha_i (a_j + \tilde{a}_j) + \sum_j \frac{1}{4} B^T B^T (a_{k+h-j} - a_j + \tilde{a}_{k+h-j} - \tilde{a}_j) \alpha_i \\ &= \sum_j \frac{1}{2} B^T \alpha_j (a_j + \tilde{a}_j) = D_\chi^+ \end{aligned}$$

where $B + B^T = 2$ has been used along with (5.18).

The situation for (5.23) is entirely analogous, with

$$B^T D_\phi^- + B^T D_\phi^- = \sum_j \frac{1}{2} B^T \alpha_j (b_j + \tilde{b}_j) = -D_\chi^- .$$

5.5.2 Proof of (5.27)

Here the final relation for momentum conservation is proven

$$2D_\phi^+ D_\phi^- - 2D_\psi^+ D_\psi^- = \Phi - \Psi .$$

Before folding, the bulk potentials are given by

$$\begin{aligned} U(u) &= \sum_{j=0}^r a_j b_j \\ X(\chi) &= \sum_{j=0}^r a_j b_{j+1} = \sum_{j=0}^r \tilde{a}_j \tilde{b}_{j+1} \\ V &= \sum_{j=0}^r \tilde{a}_j \tilde{b}_j \end{aligned}$$

and so

$$\begin{aligned} U - X &= \frac{1}{2} \sum_{i,j=0}^r \alpha_i B^T \alpha_j a_i b_j \\ V - X &= \frac{1}{2} \sum_{i,j=0}^r \alpha_i B^T \alpha_j \tilde{a}_i \tilde{b}_j \end{aligned}$$

giving the right-hand side of (5.27) as

$$\Phi - \Psi = \frac{1}{2} \sum_{i,j=0}^r \alpha_i B^T \alpha_j \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) \quad (5.28)$$

where a_i , \tilde{a}_i , b_i and \tilde{b}_i are now written in terms of the folded fields.

The left-hand side of (5.27) can be rewritten using (5.6)-(5.9) to give

$$\begin{aligned} 2D_\phi^+ D_\phi^- - 2D_\psi^+ D_\psi^- &= \sum_{i,j} \frac{1}{8} \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) \left(\alpha_i B^T B^T \alpha_j + \alpha_{k+h-i} B B \alpha_{k+h-j} \right. \\ &\quad \left. + \alpha_i B^T B \alpha_{k+h-j} + \alpha_{k+h-i} B B^T \alpha_j \right) \\ &= \sum_{i,j} \frac{1}{8} \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) \left(\alpha_i B^T (B^T + B) \alpha_j + \alpha_{k+h-i} B (B + B^T) \alpha_{k+h-j} - \alpha_i B^T B \alpha_j \right. \\ &\quad \left. - \alpha_{k+h-i} B B^T \alpha_{k+h-j} + \alpha_i B^T B \alpha_{k+h-j} + \alpha_{k+h-i} B B^T \alpha_j \right) \\ &= \frac{1}{2} \sum_{i,j} \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) \left(\alpha_i B^T \alpha_j \right) + \sum_{i,j} M_{ij} \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) \\ &= \Phi - \Psi + \sum_{i,j} M_{ij} \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) \end{aligned}$$

where use has been made of the facts that $\alpha_{k+h-i} B \alpha_{k+h-j} = \alpha_i B^T \alpha_j$ and that $B + B^T = 2$. In the last step (5.28) has been used. The quantity M_{ij} is defined as

$$M_{ij} = \frac{1}{8} (\alpha_i - \alpha_{k+h-i}) B B^T (\alpha_{k+h-j} - \alpha_j)$$

so what now remains to be shown is that

$$\sum_{i,j} M_{ij} \left(a_i b_j - \tilde{a}_i \tilde{b}_j \right) = 0. \quad (5.29)$$

First note that

$$M_{(k+h-i)j} = M_{i(k+h-j)} = -M_{ij}.$$

Using (5.18) then implies that

$$\begin{aligned} \sum_i M_{ij} a_i &= \sum_i M_{ij} a_{k+h-i} - \sum_i M_{ij} \tilde{a}_i + \sum_i M_{ij} \tilde{a}_{k+h-i} \\ &= - \sum_i M_{(k+h-i)j} a_{k+h-i} - \sum_i M_{ij} \tilde{a}_i - \sum_i M_{(k+h-i)j} \tilde{a}_{k+h-i} \\ &= - \sum_i M_{ij} a_i - 2 \sum_i M_{ij} \tilde{a}_i \\ \implies \sum_i M_{ij} a_i &= - \sum_i M_{ij} \tilde{a}_i \end{aligned}$$

while (5.19) analogously gives

$$\sum_j M_{ij} b_j = - \sum_j M_{ij} \tilde{b}_j.$$

Thus, (5.29) is true since

$$\sum_{i,j} M_{ij} a_i b_j = - \sum_{i,j} M_{ij} \tilde{a}_i b_j = + \sum_{i,j} M_{ij} \tilde{a}_i \tilde{b}_j$$

and hence

$$2D_\phi^+ D_\phi^- - 2D_\psi^+ D_\psi^- = \Phi - \Psi .$$

In conclusion, (5.26) reduces to $\dot{P} = -(\dot{D}^+ - \dot{D}^-)$ and so $P + D^+ - D^-$ is a conserved quantity, certainly when $\tilde{d} = \pm d$. Hence, energy and momentum are conserved by the folded defects. Momentum conservation in particular is a strong constraint, perhaps even implying integrability, so there is more than a hope that (5.2) with $\tilde{d} = d$ represents integrable defects for $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$.

6 Discussion

Conclusions

The main conclusion of this paper is that certain $a_r^{(1)}$ defects may be folded to give defects in the folded models of $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ with Lagrangian description in the form (5.2). There are two strong reasons to believe that such defects are integrable.

- Firstly, energy and momentum conservation, when applied to a type I $a_r^{(1)}$ defect, are in themselves enough to force the defect to be integrable [11]. It is certainly plausible that the same holds for these folded defects.
- Secondly, there is the rather simpler argument of classical scattering of solitons off the defect. This argument alone is suggestive of integrability as the solitons retain their form and so ought to conserve an infinite number of charges.

Though not the main focus, this paper also furthers the work of [9] by using the relatively simple arena of $a_r^{(1)}$ to construct solitons and breathers in $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$. The breather solutions in particular have not been published before.

There do remain a couple of loose ends at the classical level. One regards the interpretation of the auxiliary field, χ , in the folded defect. Since the algebraic constraints may be used to reduce the number of degrees of freedom of χ to that of a folded field, it seems highly likely that one may construct similar defects by starting in the folded model, i.e., without any reference to $a_r^{(1)}$. Another is that it is as yet unclear whether or not the folded defect conditions ((5.3),(5.4),(5.5)) represent a Bäcklund transformation in the folded theory- such as was found for $a_r^{(1)}$ type I defects in [2].

Future directions

Note that all ATFT defects found thus far have been purely transmitting, agreeing with the findings of Delfino, Mussardo and Simonetti [22], so one clear extension of this work would be to find the quantum transmission matrices for the folded defects. The transmission matrices for type II defects in the $a_r^{(1)}$ theories are found in [23], where the Lagrangian considered bears more

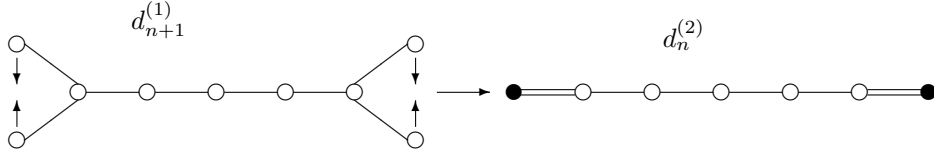


Figure 6: $d_{n+1}^{(1)} \rightarrow d_n^{(2)}$.

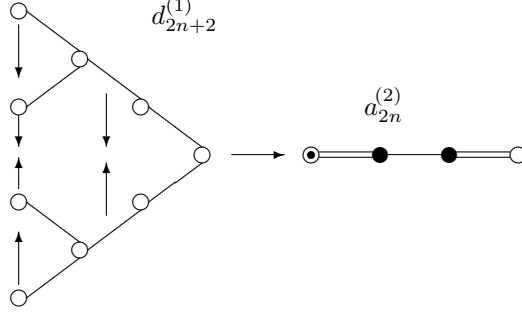


Figure 7: $d_{2n+2}^{(1)} \rightarrow a_{2n}^{(2)}$.

than a superficial similarity to (5.2). One would hope that in a similar way in which [8] presents the folded bulk S-matrices in terms of the unfolded ones, the folded transmission matrices may be found.

Another direction is to try to find defects for the other simply laced ATFTs ($d_s^{(1)}$, $e_6^{(1)}$, $e_7^{(1)}$ and $e_8^{(1)}$) which then might be folded such that all ATFTs are covered. In fact, there are possibly some implications already for $d_s^{(1)}$ defects in this paper due to the use of the non-canonical foldings $a_{2n-1}^{(1)} \rightarrow d_n^{(2)}$ and $a_{2n}^{(1)} \rightarrow a_{2n}^{(2)}$. For both of these folded theories the canonical way to fold is from a d series ATFT rather than an a series one (see figures 6 and 7), so there is the question of whether a $d_s^{(1)}$ defect ATFT, should such a thing exist, might be folded to give something of the form of (5.2).

One can also look at the possibility of using defects to find more general integrable boundary conditions. The possibility of a boundary with an auxiliary field was considered in [24]; while the paper [25] considers a defect fused to a Ghoshal–Zamolodchikov type boundary [26] in a_1 . Classically integrable boundary conditions, which happen to be highly restrictive (no free parameters), have been known for the folded theories $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$, for some time [27]. There is the possibility of fusing the defects here to such boundaries to give more general boundary conditions, though it is not immediately clear whether or not integrability would be preserved.

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<i>a priori</i> folding	<i>a posteriori</i> folding
$\phi' = {}_p B \dot{\chi} - D_\phi$	$\phi' = A \dot{\phi} + B \dot{\chi} - D_u$
$\psi' = {}_p B \dot{\chi} + D_\phi$	$\psi' = A \dot{\psi} + B \dot{\chi} + D_v$
$B^T \dot{\phi} + D_\chi^{(1)} = B^T \dot{\psi} - D_\chi^{(2)}$	$B^T \dot{\phi} + D_\chi^{(1)} = B^T \dot{\psi} - D_\chi^{(2)}$

Table 5: Comparison of the *a priori* and *a posteriori* defect conditions.

A Equivalence of *a priori* and *a posteriori* folding and ways to write D_ϕ

As is mentioned in section 5.1, there is an ambiguity in what is meant by the folding of the fused defect. There are two different points of view to consider.

1. The folding may be applied to the Lagrangian (4.10) as is done in section 5.1. This folding is *a priori* in the sense that it is done before varying the action.
2. The folding may be applied *a posteriori* to the defect equations (4.11), (4.12) and (4.13), after varying the unfolded action.

The *a posteriori* equations are given by

$$\phi' = A \dot{\phi} + B \dot{\chi} - D_u \quad (\text{A.1})$$

$$\psi' = A \dot{\psi} + B \dot{\chi} + D_v \quad (\text{A.2})$$

$$B^T \dot{\phi} + D_\chi^{(1)} = B^T \dot{\psi} - D_\chi^{(2)} . \quad (\text{A.3})$$

The comparison of (A.1), (A.2) and (A.3) to the *a priori* folded defect conditions (5.3), (5.4) and (5.5) is shown in table 5. Although the third equation on each side of table 5 matches up, there are a number of critical differences between the two sets of equations:

- The apparent self coupling found in the *a posteriori* equations and not in the *a priori* ones.
- The quantities D_u and D_v are still used in the *a posteriori* folded case even after folding has occurred. What is meant by this is that the expressions are the same as D_u and D_v respectively, but with the identifications made on the components of u and v such that D_u and D_v are written with ϕ and ψ , respectively, in the exponents. These quantities do not match D_ϕ and D_ψ , as the meaning of the gradient is changed by folding. In summary: $u \rightarrow \phi \not\Rightarrow D_u \rightarrow D_\phi$.
- Perhaps the most fundamental difference is in the number of equations obtained when considered in component form. Whilst the variation of the action required to obtain the *a posteriori* equations is always done on the full $a_r^{(1)}$ root space, the first two equations of the *a priori* case are obtained by varying in the folded root space- or with respect to only symmetric configurations of the $a_r^{(1)}$ root space. Thus, there are always fewer equations of motion in the *a priori* case- so it should be the true that if the folding processes are not completely equivalent, then at least the *a posteriori* case will contain the *a priori* case and possibly some extra information.

- The presence of the subscript p before the $B\dot{\chi}$ in the first two *a priori* equations is there to emphasise that the $B\dot{\chi}$ found in those equations only makes sense in components when projected onto the folded root/weight space. It is valid to consider $(\alpha_i + \alpha_{k+h-i}) B\dot{\chi}$ since $\alpha_i + \alpha_{k+h-i}$ lies in the folded root space, but it is not valid to consider $(\alpha_i - \alpha_{k+h-i}) B\dot{\chi}$ since $\alpha_i - \alpha_{k+h-i}$ does not lie in the folded root space. For the *a posteriori* case both projections may be considered as they both lie on the unfolded $a_r^{(1)}$ root space.

By first tackling the form of D_ϕ and D_ψ it will become possible to compare each pair of equations above in turn.

A.1 Gradients of the defect potential

In section 2 it is seen that despite all of the possible bivalent foldings of $a_r^{(1)}$ having major differences, all of the folded roots did take the universal form of $\frac{\alpha_i + \alpha_{k+h-i}}{2}$. It would be helpful to be able to write the gradients of the potential in a universal way, rather than having to consider $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$ separately.

First consider the folded field ϕ written in a different manner to the standard one of section 2. For $c_n^{(1)}$ and $a_{2n}^{(2)}$ this is

$$\phi = \sum_{j=\frac{k}{2}+1}^{\frac{k}{2}+n} \frac{\alpha_j + \alpha_{k+h-j}}{2} \tilde{\phi}_j$$

with $\tilde{\phi}_{\frac{k}{2}+i} = \phi_i$.

Similarly, if the folded theory is $d_n^{(2)}$ the field is

$$\phi = \sum_{j=\frac{k+1}{2}+1}^{\frac{k+1}{2}+n-1} \frac{\alpha_j + \alpha_{k+h-j}}{2} \tilde{\phi}_j$$

with $\tilde{\phi}_{\frac{k+1}{2}+i} = \phi_i$.

The reason for this relabelling is to make contact to the components u_i and u_{k+h-i} in the defect potential D . Folding is, in all but one instance, achieved by setting $u_i = u_{k+h-i} = \frac{\tilde{\phi}_i}{2}$ (the exception being that $u_{\frac{k}{2}+n} = \tilde{\phi}_{\frac{k}{2}+n}$ in $c_n^{(1)}$) and so the derivative of D with respect to the component $\tilde{\phi}_i$ is given by

$$D_{\tilde{\phi}_i} = \frac{1}{2} (D_{u_i} + D_{u_{k+h-i}}) \quad (\text{A.4})$$

which is true for all of the identifications, including the exception mentioned above. Given this, $D_{\tilde{\phi}_i}$ may be found explicitly using the relations (4.20). The u dependence only appears in $D^{(1)}$, and it will be necessary to split the positive and negative helicity parts, so the quantities of interest are

$$\begin{aligned} D^{(1)+} &= d \sum_{j=0}^r e^{\frac{1}{2} \alpha_j (B^T u + B \chi)} = d \sum_{j=0}^r e^{u_j - u_{j+1} + \chi_j - \chi_{j-1}} = \sum_{j=0}^r a_j \\ D^{(1)-} &= \frac{1}{d} \sum_{j=0}^r e^{\frac{1}{2} \alpha_j B(u - \chi)} = \frac{1}{d} \sum_{j=0}^r e^{u_j - u_{j-1} - \chi_j + \chi_{j-1}} = \sum_{j=0}^r b_j \end{aligned}$$

with the identifications

$$a_j = e^{u_j - u_{j+1} + \chi_j - \chi_{j-1}} \quad \text{and} \quad b_j = e^{u_j - u_{j-1} - \chi_j + \chi_{j-1}} .$$

Thus, written in this compact notation it is seen that

$$\begin{aligned} D_{u_i}^+ &= a_i - a_{i-1} & , & & D_{u_{k+h-i}}^+ &= a_{k+h-i} - a_{k+h-1-i} \\ D_{u_i}^- &= b_i - b_{i+1} & , & & D_{u_{k+h-i}}^- &= b_{k+h-i} - b_{k+h+1-i} \end{aligned}$$

so, explicitly, (A.4) may be rewritten as

$$\begin{aligned} D_{\tilde{\phi}_i}^+ &= \frac{1}{2} (a_i - a_{i-1} + a_{k+h-i} - a_{k+h-1-i}) \\ D_{\tilde{\phi}_i}^- &= \frac{1}{2} (b_i - b_{i+1} + b_{k+h-i} - b_{k+h+1-i}) \end{aligned} \tag{A.5}$$

which is true of all the bivalent foldings.

One must now consider how D_ϕ is related to $D_{\tilde{\phi}_i}$ and hope that it may be written in a similarly universal form. Consider first the unfolded case of D_u . As mentioned in section 2, u may be written as

$$u = \sum_{j=0}^r u_j \alpha_j$$

with one of the components, u_q , set to zero. The reciprocal basis is then just the set of fundamental weights $\{\lambda_i\}$ that has $\lambda_q = 0$. Thus, the gradient with respect to u may be written

$$\nabla_u = \sum_{j=0}^r \lambda_j \frac{\partial}{\partial u_j}$$

and so

$$D_u = \sum_{j=0}^r \lambda_j D_{u_j} .$$

For the folded field it is a similar picture. If ϕ is in $c_n^{(1)}$ or $a_{2n}^{(2)}$ the field is

$$\phi = \sum_{j=1}^n \phi_j \alpha'_j$$

and so

$$D_\phi = \sum_{j=1}^n \lambda'_j D_{\phi_j} \tag{A.6}$$

where $\lambda'_i \cdot \alpha'_j = \delta_{ij}$ for $i, j = 1, \dots, n$ ($\lambda'_0 = 0$), meaning that $\{\lambda'_i\}$ are the fundamental weights of the folded theory. If ϕ is instead in $d_n^{(2)}$, the same analysis holds but with an upper limit to the sum of $n - 1$ rather than n .

One can use the expression for α'_i found in section 2 to write λ'_i in terms of the weights of $a_r^{(1)}$, with all of the folded theories; $c_n^{(1)}$, $d_n^{(2)}$ and $a_{2n}^{(2)}$; having differences resulting in three different possible expressions for D_ϕ . These expressions are not written here, as it can be shown that they are all encompassed by the universal form

$$D_\phi = \frac{1}{2} \sum_{j=0}^{h-1} (\lambda_j + \lambda_{k+h-j}) D_{\tilde{\phi}_j} . \quad (\text{A.7})$$

The expression (A.7) counts all the non-zero terms twice apart from the $j = \frac{k}{2} + n$ case of $c_n^{(1)}$. The results (A.5) can now be used, resulting in

$$D_\phi^+ = \frac{1}{4} \sum_{j=0}^{h-1} (\lambda_j + \lambda_{k+h-j}) (a_i - a_{i-1} + a_{k+h-i} - a_{k+h-1-i}) \quad (\text{A.8})$$

$$D_\phi^- = \frac{1}{4} \sum_{j=0}^{h-1} (\lambda_j + \lambda_{k+h-j}) (b_i - b_{i+1} + b_{k+h-i} - b_{k+h+1-i}) . \quad (\text{A.9})$$

While (A.5) is useful in comparing the *a priori* and *a posteriori* defect equations, the form of (A.7) is not easy to work with when considering the momentum conservation of the folded defect, which requires, among other things, the quadratic quantity D_ϕ^2 to be calculated. Working out the inner products between the weights of $a_r^{(1)}$, which would need to be done for D_ϕ^2 , can be done on a case by case basis, but it is not clear how to do this in general so an easier to use form of D_ϕ is a necessity.

A.1.1 A more useful way to write D_ϕ

For the unfolded fields, the gradients of $D^{(1)}$ are already given in section 4.2.2 in an alternative form. The gradients of D can similarly be written

$$\begin{aligned} D_u^+ &= \sum_j \frac{1}{2} B \alpha_j a_j & D_u^- &= \sum_j \frac{1}{2} B^T \alpha_j b_j \\ D_v^+ &= \sum_j \frac{1}{2} B \alpha_j \tilde{a}_j & D_v^- &= \sum_j \frac{1}{2} B^T \alpha_j \tilde{b}_j \\ D_\chi^+ &= \sum_j \frac{1}{2} B^T \alpha_j (a_j + \tilde{a}_j) & D_\chi^- &= - \sum_j \frac{1}{2} B^T \alpha_j (b_j + \tilde{b}_j) . \end{aligned} \quad (\text{A.10})$$

However, it is unclear from (4.20) what the analogous expressions for the gradients of D with respect to the folded fields ϕ and ψ are. Here the form of D_ϕ is proposed to be

$$D_\phi^+ = \sum_{j=0}^{h-1} \frac{1}{4} (B \alpha_j + B^T \alpha_{k+h-j}) a_j \quad (\text{A.11})$$

$$D_\phi^- = \sum_{j=0}^{h-1} \frac{1}{4} (B^T \alpha_j + B \alpha_{k+h-j}) b_j . \quad (\text{A.12})$$

The equations (A.11) and (A.12) are now shown to be true. The form of B must be used from

(4.2) along with its transpose, i.e.,

$$B = 2 \sum_{j=0}^r (\lambda_j - \lambda_{j+1}) \lambda_j^T$$

$$B^T = 2 \sum_{j=0}^r (\lambda_j - \lambda_{j-1}) \lambda_j^T$$

which results in

$$B\alpha_i + B^T\alpha_{k+h-i} = 2(\lambda_i - \lambda_{i+1} + \lambda_{k+h-i} - \lambda_{k+h-i-1}) \quad (\text{A.13})$$

$$B^T\alpha_i + B\alpha_{k+h-i} = 2(\lambda_i - \lambda_{i-1} + \lambda_{k+h-i} - \lambda_{k+h-i+1}) . \quad (\text{A.14})$$

The first of these identities, (A.13) appears in D_ϕ^+ , so reordering sums in two ways gives

$$D_\phi^+ = \frac{1}{2} \sum_{j=0}^{h-1} \lambda_j (a_j - a_{j-1} + a_{k+h-j} - a_{k+h-j+1})$$

$$D_\phi^+ = \frac{1}{2} \sum_{j=0}^{h-1} \lambda_{h+k-j} (a_j - a_{j-1} + a_{k+h-j} - a_{k+h-j+1})$$

which may be combined to give

$$D_\phi^+ = \frac{1}{4} \sum_{j=0}^{h-1} (\lambda_j + \lambda_{k+h-j}) (a_j - a_{j-1} + a_{k+h-j} - a_{k+h-j+1})$$

which matches (A.8), thus proving that (A.11) is correct. In exactly the same way (A.14) can be used to show that (A.12) matches (A.9).

Having these universal forms for the gradients will allow momentum conservation to be shown for all of the folded theories using the same argument. Everything that has been shown here for D_ϕ has exact equivalents for D_ψ , the difference being that $\phi \rightarrow \psi$ while $a_i \rightarrow \tilde{a}_i$, $b_i \rightarrow \tilde{b}_i$; hence, there is no need to prove anything for D_ψ separately.

All of the information needed to compare the *a priori* defect conditions ((5.3), (5.4) and (5.5)) to the *a posteriori* defect conditions ((A.1), (A.2) and (A.3)) is given by (A.7) with (A.4).

A.2 Comparison of the *a priori* and *a posteriori* defect conditions

The first equations to compare are those obtained from the Euler-Lagrange equation for the bulk field on the left (ϕ for the *a priori* case; u for the *a posteriori* case), i.e., (5.3) and (A.1).

$$\phi' = {}_p B \dot{\chi} - D_\phi \quad | \quad \phi' = A \dot{\phi} + B \dot{\chi} - D_u .$$

There are clear differences between the equations, particularly as regards the number of equations represented in component form. For both types of folding to match it should at least be the case that (A.1) has the same projection onto the folded root space as (5.3). Projection can

be made onto the folded root space by taking the inner product with $\alpha_i + \alpha_{k+h-i}$; the left-hand sides match up so what needs to be compared is

$$\begin{aligned}
(\alpha_i + \alpha_{k+h-i}) \cdot \text{RHS}(5.3) &\longrightarrow (\alpha_i + \alpha_{k+h-i})_p B\dot{\chi} - (\alpha_i + \alpha_{k+h-i}) \cdot D\phi \\
&= (\alpha_i + \alpha_{k+h-i}) B\dot{\chi} - \frac{1}{2} \sum_j (\alpha_i + \alpha_{k+h-i}) \cdot (\lambda_j + \lambda_{k+h-j}) (D_{u_j} + D_{u_{k+h-j}}) \\
&= (\alpha_i + \alpha_{k+h-i}) B\dot{\chi} - D_{u_i} - D_{u_{k+h-i}}
\end{aligned}$$

with

$$\begin{aligned}
(\alpha_i + \alpha_{k+h-i}) \cdot \text{RHS}(A.1) &\longrightarrow (\alpha_i + \alpha_{k+h-i}) A\dot{\phi} + (\alpha_i + \alpha_{k+h-i}) B\dot{\chi} - (\alpha_i + \alpha_{k+h-i}) \cdot D_u \\
&= (\alpha_i + \alpha_{k+h-i}) B\dot{\chi} - \sum_j (\alpha_i + \alpha_{k+h-i}) \cdot \lambda_j D_{u_j} \\
&= (\alpha_i + \alpha_{k+h-i}) B\dot{\chi} - D_{u_i} - D_{u_{k+h-i}} .
\end{aligned}$$

These match, but there could be extra information contained in (A.1) in the part of the $a_r^{(1)}$ root space which is not on the folded space. This extra information may actually be contained elsewhere, perhaps in the time derivatives of the algebraic constraints (5.12) found in the χ equation of motion. Some evidence for this supposition is provided for the $a_2^{(2)}$ case in appendix B.

The equations (5.4) and (A.2) give a completely similar picture to the above analysis so there is no need to consider them explicitly.

Also clear is that the equations (5.5) and (A.3) are exactly the same, so the χ equation of motion is unchanged by the different folding processes. Examining the components of (5.5) a set of algebraic constraints is found:

$$\begin{aligned}
\alpha_i \cdot (B^T \dot{\phi} - B^T \dot{\psi}) &= \alpha_i \cdot (-D_\chi^{(1)} - D_\chi^{(2)}) = -\alpha_i \cdot D_\chi \\
\implies \dot{\phi}_i - \dot{\phi}_{i+1} - \dot{\psi}_i + \dot{\psi}_{i+1} &= -D_{\chi_i}
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{k+h-1-i} \cdot (B^T \dot{\phi} - B^T \dot{\psi}) &= -\alpha_{k+h-1-i} \cdot D_\chi \\
\implies -\dot{\phi}_i + \dot{\phi}_{i+1} + \dot{\psi}_i - \dot{\psi}_{i+1} &= -D_{\chi_{k+h-1-i}}
\end{aligned}$$

and consequently

$$D_{\chi_i} + D_{\chi_{k+h-1-i}} = 0 \tag{A.15}$$

where $D = D^{(1)} + D^{(2)}$. In this form the constraints are valid for all of the bivalent foldings. It is worth checking how these constraints appear in terms of a_i, \tilde{a}_i, b_i and \tilde{b}_i which involves

$$D_{\chi_i} = a_i - a_{i+1} + \tilde{a}_i - \tilde{a}_{i+1} - b_i + b_{i+1} - \tilde{b}_i + \tilde{b}_{i+1} .$$

For each folding the number of constraints is

- $c_n^{(1)}$ the $n+1$ relations become n as the relation for $i = \frac{k}{2} + n$ is a trivial identity $0 = 0$. The remaining n relations are not all linearly independent but $n-1$ of them are- so there are $n-1$ algebraic constraints- reducing the number of degrees of freedom of χ from $2n-1$ down to n .

- $d_n^{(2)}$: n of the $n + 1$ relations are independent so χ has $(2n - 1) - n = n - 1$ degrees of freedom remaining.
- $a_{2n}^{(2)}$: n independent constraints so χ has $2n - n = n$ degrees of freedom remaining

Using the constraints, degrees of freedom in the auxiliary field χ may be removed such that the root space dimensionality of χ is the same in each case as the dimensionality of the folded fields ϕ and ψ .

In each case the relations (A.15) may be written in a more useful form

$$a_i - a_{k+h-i} + \tilde{a}_i - \tilde{a}_{k+h-i} - b_i + b_{k+h-i} - \tilde{b}_i + \tilde{b}_{k+h-i} = 0 .$$

Since each a_i is a positive helicity term and each b_i has negative helicity, the algebraic constraints generally mix helicity which is an undesirable property. The remedy to this problem is provided in section 5.3 which examines the passing of solitons through the defect.

B The folded defect in $a_2^{(2)}$

This appendix deals with the particular case of $a_2^{(2)}$, also known as the T̃ițeica or the Bullough–Dodd model, when considering the defect of section 5. It is taken that $k = 0$ in terms of the bivalent foldings, and also that $\tilde{d} = d$ to give the non-trivial defect, so in components the fields are

$$\begin{aligned}\phi &= \frac{\alpha_1 + \alpha_2}{2} \phi \\ \psi &= \frac{\alpha_1 + \alpha_2}{2} \psi \\ \chi &= \alpha_1 \chi_1 + \alpha_2 \chi_2 .\end{aligned}$$

The ambiguity over the meaning of ϕ and ψ will disappear since everything will be expressed in terms of components henceforth. In components the Lagrangian (5.2) is then

$$\mathcal{L} = \theta(-x)\mathcal{L}_\phi + \theta(x)\mathcal{L}_\psi + \delta(x)((\phi - \psi)\dot{\chi}_2 - D) \quad (\text{B.1})$$

with

$$\begin{aligned}\mathcal{L}_\phi &= \frac{1}{4}\dot{\phi}\dot{\phi} - \frac{1}{4}\phi'\phi' - \Phi \\ \Phi &= e^{-\phi} + 2e^{\frac{\phi}{2}}\end{aligned}$$

and an entirely analogous picture for \mathcal{L}_ψ .

The defect potential is given by

$$\begin{aligned}D &= d \left(e^{-\frac{\phi}{2} - \chi_2} + e^{\frac{\phi}{2} - \chi_1 + \chi_2} + e^{-\frac{\psi}{2} - \chi_2} + e^{\frac{\psi}{2} - \chi_1 + \chi_2} + 2e^{\chi_1} \right) \\ &\quad + \frac{1}{d} \left(e^{-\frac{\phi}{2} + \chi_2} + e^{\frac{\phi}{2} - \chi_1} + e^{-\frac{\psi}{2} + \chi_2} + e^{\frac{\psi}{2} - \chi_1} + 2e^{\chi_1 - \chi_2} \right) .\end{aligned} \quad (\text{B.2})$$

The defect equations of (B.1) at $x = 0$ are

$$\phi' = 2\dot{\chi}_2 - 2D_\phi \quad (\text{B.3})$$

$$D_{\chi_1} = 0 \quad (\text{B.4})$$

$$\dot{\phi} - \dot{\psi} = -D_{\chi_2} \quad (\text{B.5})$$

$$\psi' = 2\dot{\chi}_2 + 2D_\psi. \quad (\text{B.6})$$

The first thing to note now is the algebraic constraint (B.4). Since $\tilde{d} = d$ it is expected that the algebraic constraint does not mix helicities (i.e., $D_{\chi_1}^+ = D_{\chi_1}^- = 0$) and indeed that is the case. Explicitly, using (B.2), the situation is

$$D_{\chi_1} = (d + d^{-1}e^{-\chi_2}) \left(2e^{\chi_1} - e^{\frac{\phi}{2} - \chi_1 + \chi_2} - e^{\frac{\psi}{2} - \chi_1 + \chi_2} \right) = 0$$

so it is clear that $D_{\chi_1}^+$ and $D_{\chi_1}^-$ vanish separately. The algebraic constraint may be written as

$$e^{2\chi_1} = \frac{1}{2}e^{\chi_2} \left(e^{\frac{\phi}{2}} + e^{\frac{\psi}{2}} \right) \quad (\text{B.7})$$

meaning that the χ_1 degree of freedom may be removed such that the auxiliary field has just one degree of freedom, χ_2 . One aspect of using (B.7) to remove the χ_1 terms in the exponentials in the potential D is that it results in terms coupling ϕ to ψ directly- this situation is also seen in the type II defects of [3].

Also notable about the potential is that

$$D_{\chi_2}^+ = 2D_\phi^+ + 2D_\psi^+ \quad (\text{B.8})$$

$$D_{\chi_2}^- = -2D_\phi^- - 2D_\psi^- \quad (\text{B.9})$$

where the first relation (B.8) is seen naturally in the form of (B.2); but the second, (B.9), requires use of the algebraic constraint $D_{\chi_1}^- = 0$ (or, alternatively, (B.7)).

Due to (B.8) and (B.9), the defect condition (B.5) may be rewritten as

$$\dot{\phi} + 2D_\phi^+ - 2D_\phi^- = \dot{\psi} - 2D_\psi^+ + 2D_\psi^-. \quad (\text{B.10})$$

Energy and momentum conservation are now provided in components

B.1 Energy conservation

The situation here is precisely that of section 5.4 in $a_2^{(2)}$. In components the energy conservation equation is

$$\dot{E} = \frac{1}{2}\dot{\phi}\phi' - \frac{1}{2}\dot{\psi}\psi'.$$

Using (B.3), (B.6) and then (B.5) gives

$$\begin{aligned} \dot{E} &= \frac{1}{2}\dot{\phi}\phi' - \frac{1}{2}\dot{\psi}\psi' \\ &= -\dot{\phi}D_\phi - \dot{\psi}D_\psi - \dot{\chi}_1D_{\chi_1} - \dot{\chi}_2D_{\chi_2} \\ &= -\dot{D} \end{aligned}$$

with zero being added in the form of $-\dot{\chi}_1D_{\chi_1}$. Thus $E + D$ is conserved as required.

B.2 Momentum conservation

The $a_2^{(2)}$ component form version of the argument of section 5.5 is

$$\dot{P} = \frac{1}{4} \left(\dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi} + \dot{\phi}'\dot{\phi}' - \dot{\psi}'\dot{\psi}' \right) - \Phi + \Psi .$$

Squaring (B.3) and (B.6) give

$$\begin{aligned} \frac{1}{4} (\dot{\phi}'\dot{\phi}' - \dot{\psi}'\dot{\psi}') &= -2\dot{\chi}_2 (D_\phi + D_\psi) + D_\phi^2 - D_\psi^2 \\ &= -\dot{\chi}_2 (D_{\chi_2}^+ - D_{\chi_2}^-) - \dot{\chi}_1 (D_{\chi_1}^+ - D_{\chi_1}^-) + D_\phi^2 - D_\psi^2 \end{aligned}$$

by (B.8), (B.9) and the fact that $D_{\chi_1}^+ - D_{\chi_1}^- = 0$. Similarly, squaring both sides of (B.10) and equating results in

$$\frac{1}{4} (\dot{\phi}\dot{\phi} - \dot{\psi}\dot{\psi}) = -\dot{\phi} (D_\phi^+ - D_\phi^-) - \dot{\psi} (D_\psi^+ - D_\psi^-) + (D_\psi^+ - D_\psi^-)^2 - (D_\phi^+ - D_\phi^-)^2$$

and thus the momentum conservation equation reduces to

$$\dot{P} = - (\dot{D}^+ - \dot{D}^-) + 4D_\phi^+ D_\phi^- - 4D_\psi^+ D_\psi^- - \Phi + \Psi .$$

It is seen now that this is of the expected form since

$$\begin{aligned} 4D_\phi^+ D_\phi^- - 4D_\psi^+ D_\psi^- &= e^{-\phi} - e^{-\psi} + (e^\phi - e^\psi) e^{-2\chi_1 + \chi_2} \\ &= e^{-\phi} - e^{-\psi} + (e^\phi - e^\psi) \frac{2}{e^{\frac{\phi}{2}} + e^{\frac{\psi}{2}}} \\ &= e^{-\phi} + 2e^{\frac{\phi}{2}} - e^{-\psi} - 2e^{\frac{\psi}{2}} \\ &= \Phi - \Psi . \end{aligned}$$

Hence, $P + D^+ - D^-$ is conserved. This is known already from section 5.5 so is nothing other than an explicit verification.

B.3 Algebraic constraints and *a posteriori* folding

This $a_2^{(2)}$ case has been examined from the point of view of *a priori* folding thus far, but there is the concern that the $a_2^{(2)}$ versions of (A.1) and (A.2) may contain extra information not found in the *a priori* case. Indeed extra defect conditions are found which are

$$(\alpha_1 - \alpha_2) \cdot (\text{A.1}) \rightarrow 0 = -\dot{\phi} + 4\dot{\chi}_1 - 2\dot{\chi}_2 - D_{u_1} + D_{u_2} \quad (\text{B.11})$$

$$(\alpha_1 - \alpha_2) \cdot (\text{A.2}) \rightarrow 0 = -\dot{\psi} + 4\dot{\chi}_1 - 2\dot{\chi}_2 + D_{v_1} - D_{v_2} . \quad (\text{B.12})$$

An apparently extra condition may also be obtained in the *a priori* (and *a posteriori*) folded case from the algebraic constraint (B.7). For (B.7) to be true for all times it must be that case that

$$\frac{d}{dt} (2e^{2\chi_1 - \chi_2}) = \frac{d}{dt} (e^{\frac{\phi}{2}} + e^{\frac{\psi}{2}})$$

and so

$$\frac{1}{2} (4\dot{\chi}_1 - 2\dot{\chi}_2) (2e^{2\chi_1 - \chi_2}) = \frac{1}{2} \left(\dot{\phi} e^{\frac{\phi}{2}} + \dot{\psi} e^{\frac{\psi}{2}} \right) .$$

Applying (B.7) to this results in

$$\left(-\dot{\phi} + 4\dot{\chi}_1 - 2\dot{\chi}_2 \right) e^{\frac{\phi}{2}} + \left(-\dot{\psi} + 4\dot{\chi}_1 - 2\dot{\chi}_2 \right) e^{\frac{\psi}{2}} = 0 . \quad (\text{B.13})$$

So the situation is that (B.13) is true in the *a priori* folded case and this might just contain the same information as (B.11) and (B.12). What is certainly true is that (B.13) can be verified by means of (B.11) and (B.12); applying these to (B.13) gives

$$\begin{aligned} \left(-\dot{\phi} + 4\dot{\chi}_1 - 2\dot{\chi}_2 \right) e^{\frac{\phi}{2}} + \left(-\dot{\psi} + 4\dot{\chi}_1 - 2\dot{\chi}_2 \right) e^{\frac{\psi}{2}} &= (D_{u_1} - D_{u_2}) e^{\frac{\phi}{2}} + (-D_{v_1} + D_{v_2}) e^{\frac{\psi}{2}} \\ &= 2de^{-\chi_1 + \chi_2} \left(2e^{2\chi_1 - \chi_2} \left(e^{\frac{\phi}{2}} - e^{\frac{\psi}{2}} \right) - e^{\phi} + e^{\psi} \right) \\ &\quad + d^{-1}e^{-\chi_1} \left(-2e^{2\chi_1 - \chi_2} \left(e^{\frac{\phi}{2}} - e^{\frac{\psi}{2}} \right) + e^{\phi} - e^{\psi} \right) \\ &= 0 \end{aligned}$$

where the algebraic constraint (B.7) is used in the final step.

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